Symmetries of Spheres

Siddhartha Gadgil

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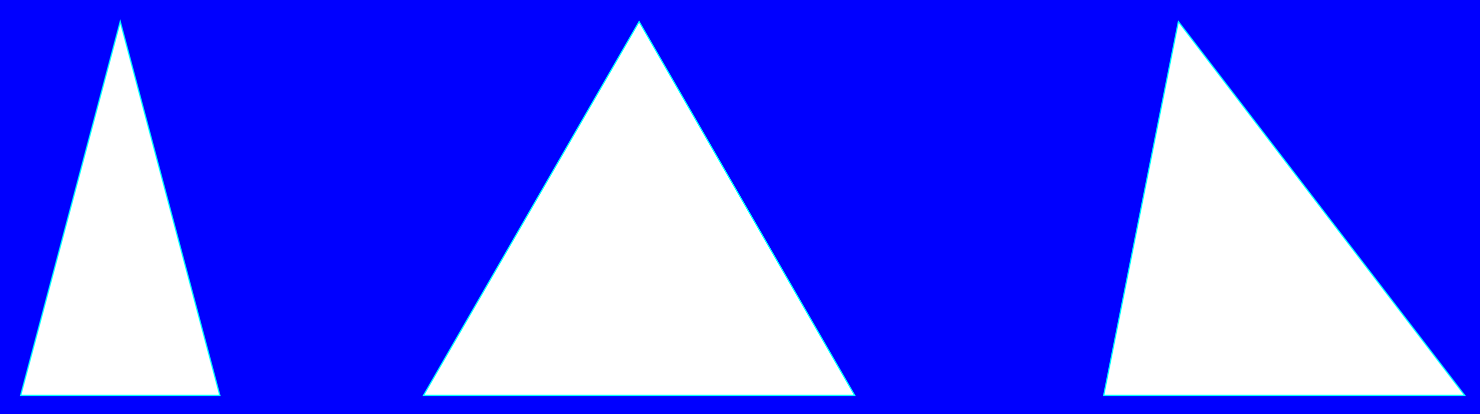

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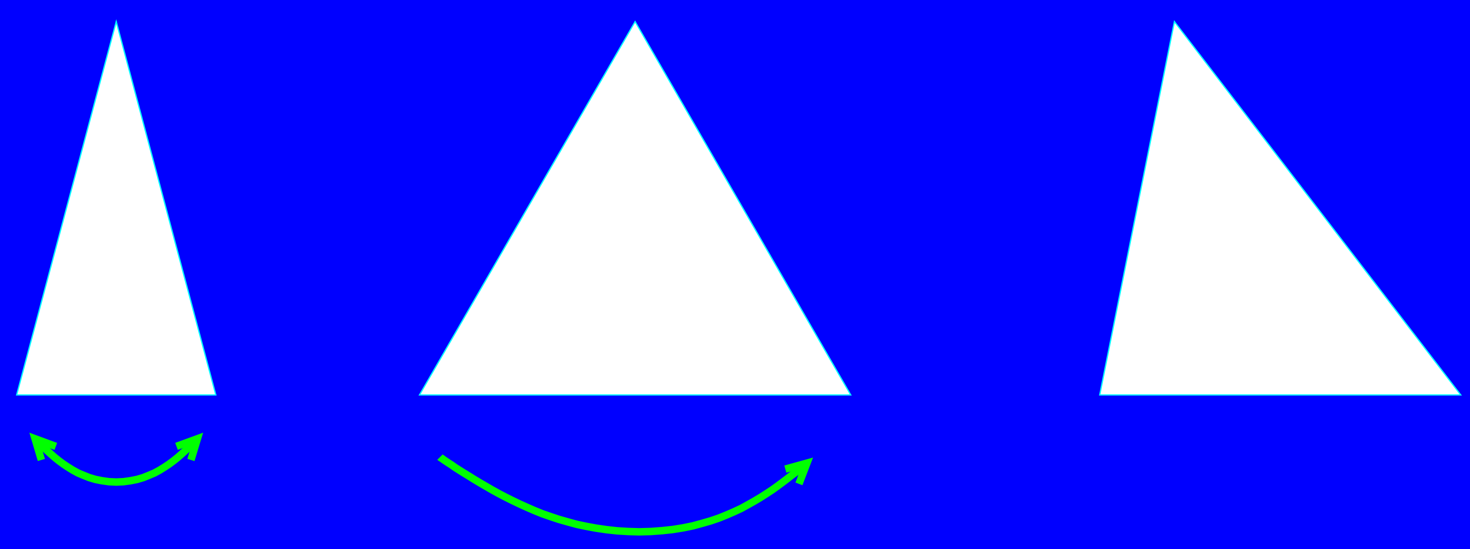

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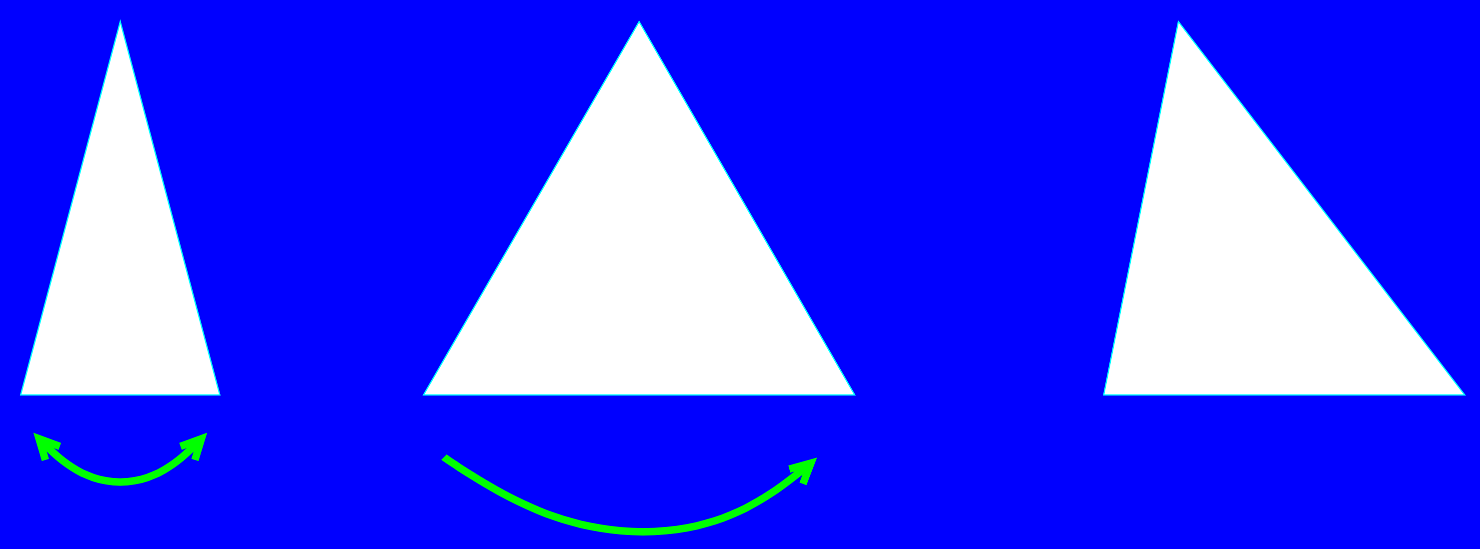

- A symmetry of a triangle is a transformation taking vertices to vertices and preserving lengths.
- In general a symmetry is a transformation preserving the appropriate structures (for example distances).

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- More generally, we say the *groups* $\langle S \rangle$ and $\langle T \rangle$ are equivalent if the sets $\{S, S^2, \dots S^k\}$ and $\{T, T^2, \dots T^k\}$ are equal after a change of co-ordinates. For example, if S is the rotation of the plane by $\pi/3$ and T is the rotation by $2\pi/3$, $\langle S \rangle$ and $\langle T \rangle$ are equivalent.

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- We shall consider *smooth* T (not necessarily preserving distances). We allow smooth changes of co-ordinates.

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- In dimensions 5 and above, this is false. But symmetries of S^n up to equivalence are classified.
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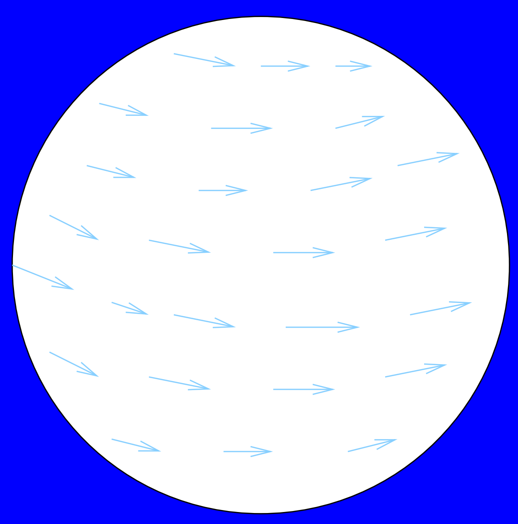
- We shall consider the case when $n = 3$ and T has no fixed points.

The Poincaré-Hopf theorem:



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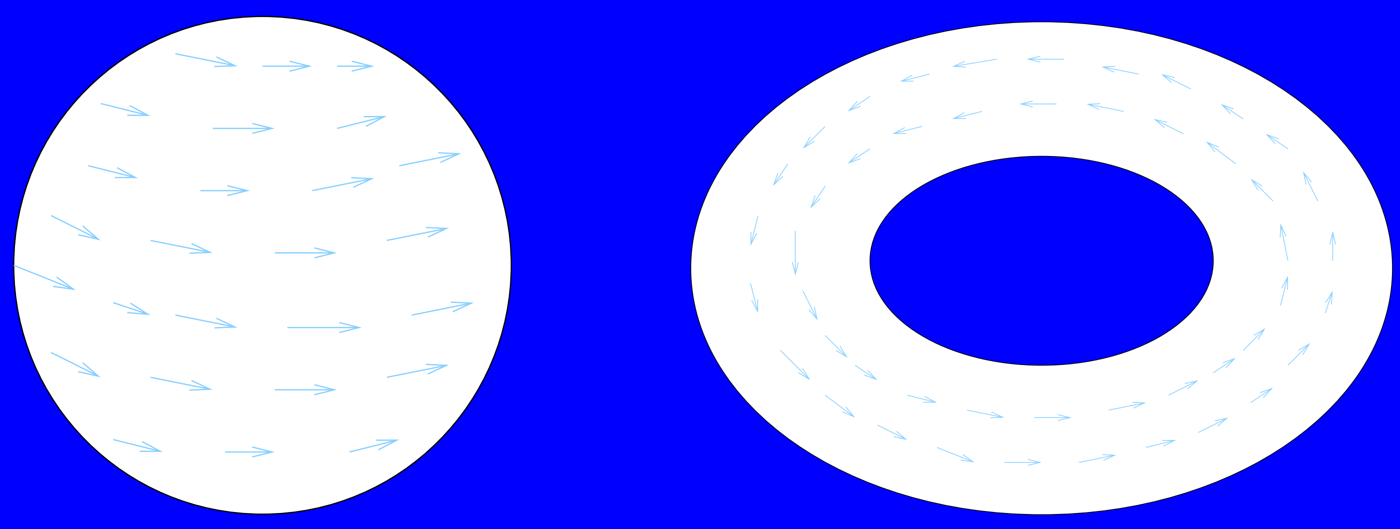

- Water flowing smoothly on a sphere must be stationary at some point.
- This is not so for water flowing on a torus - the surface of a doughnut.

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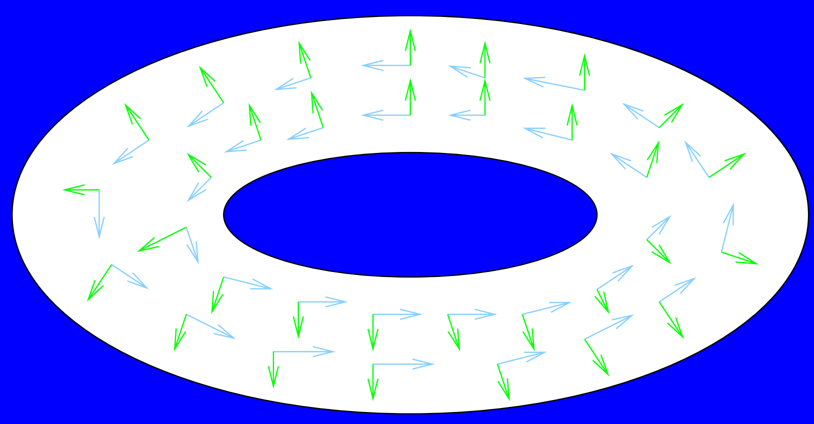
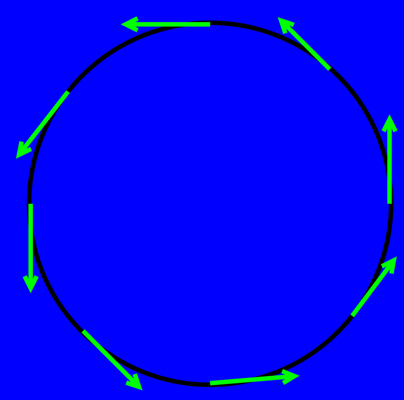
- This is a theorem in topology because it does not depend on the sphere being round - it is also true for the surface of an egg.
- Topological properties are those that are preserved by any smooth transformation (or any continuous transformation).
- Locally a sphere and a torus are the same. Topological properties are the global properties.

Framings





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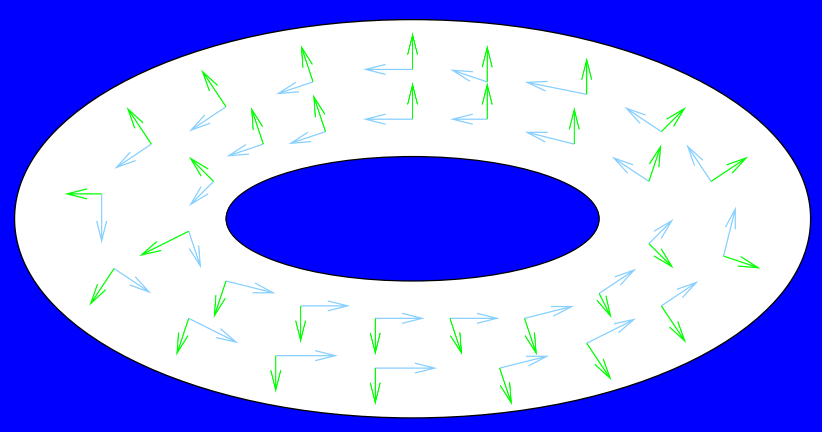

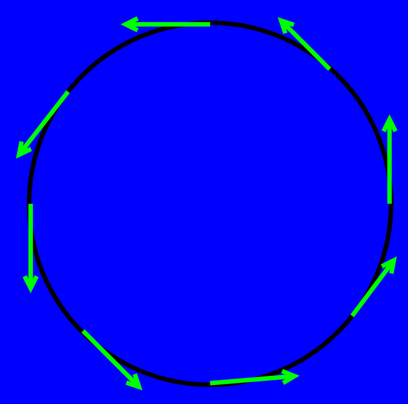

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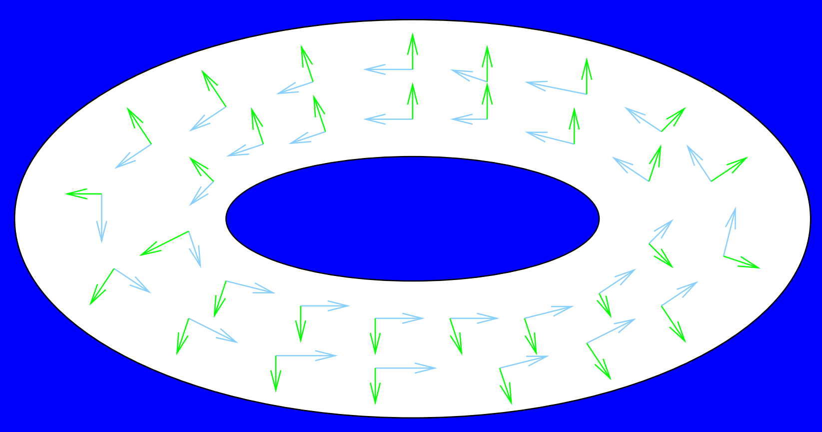

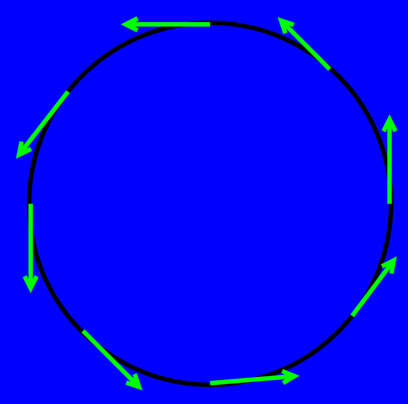

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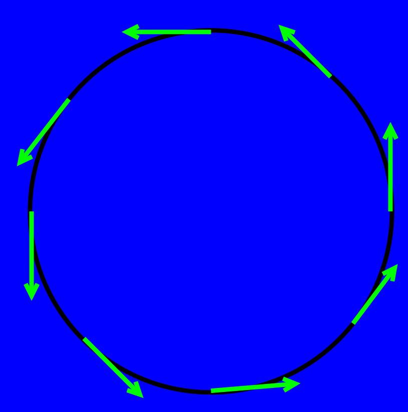
- A *framing* of a k-dimensional space M is a collection of k smooth unit vector fields tangent to M that are mutually perpendicular.
- Two framings are equivalent if we can continuously deform one to the other.

Unit vector fields on S^1



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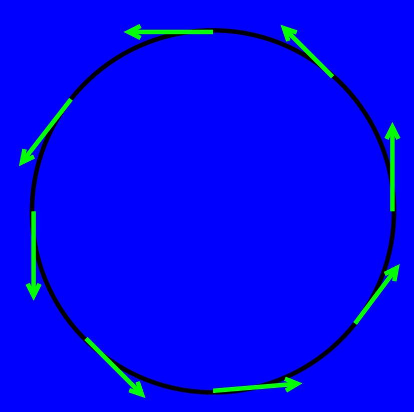

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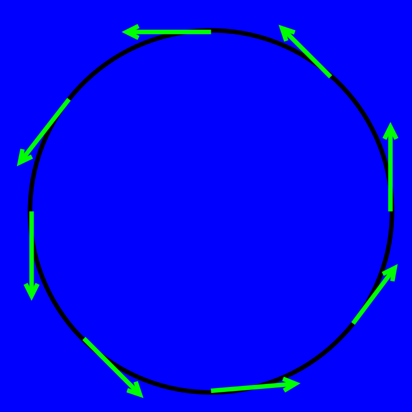

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- Two unit vector fields on S^1 are equivalent if and only if the degrees of the corresponding maps are equal.

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$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1;\ \boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{jk} = \boldsymbol{i}, \boldsymbol{ki} = \boldsymbol{j}$$

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- For $z = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$,
  - $\bar{z} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$
  - $|z|^2 = z\bar{z} = a^2 + b^2 + c^2 + d^2$
  - z is a *unit quaternion* if $|z| = 1$
  - z is *purely imaginary* if $a = 0$.

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- Thus, we can essentially identify rotations in dimension 3 with the unit quaternions S^3 (more precisely with $S^3/\pm 1$).
- Similarly, using $R_{zz'} : w \to zwz'$, we get an identification of rigid body motions of $\mathbb{R}^4$ with $(S^3 \times S^3)/\pm(1,1)$.

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- We have a similar framing for S^7. These are the only spheres with framings (Milnor-Bott, Kervaire, J.F.Adams).
- Consequently, the only dimensions in which we have a bilinear product $\mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ with $u, v \neq 0 \implies uv \neq 0$ are 1, 2, 4 and 8.

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- Given two *positive* framings ξ and ζ of S^3 and a point $p \in S^3$, we get two orthonormal bases $\xi(p)$ and $\zeta(p)$ of the tangent space of S^3 at p. These differ by a rotation, hence an element of S^3.

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- Thus we get a function $f : S^3 \to S^3$. Its degree (an integer) is called the difference between the two framings.
- We regard 0 as corresponding to the left invariant framing. Then framings of S^3 correspond to the integers $\mathbb{Z}$ and the right invariant framing corresponds to 1.

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- This is related to the other fundamental question regarding symmetries of S^3:
- **Question 2:** Is T equivalent to a rigid body motion?

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- Thus, the *congruence class* $\mathfrak{F}(T)$ of an equivariant framing is an invariant of the group $\langle T \rangle$.

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  Theorem. *$\langle L_{p,q} \rangle$ is equivalent to $\langle L_{p,q'} \rangle$ if and only if $q' \equiv \pm q (mod\ p)$ or $qq' \equiv 1 (mod\ p)$.*
- Our proof has the advantage of being naturally related to the *exceptional isomorphisms* (relating rotations in dimensions 3 and 4 to S^3) which play a central role in Topology and Gauge theory in dimensions 3 and 4.

"Most people, even some scientists, think that mathematics applies because you learn Theorem Three and Theorem Three somehow explains the laws of nature. This does not happen even in science fiction novels, it is pure fantasy. The results of mathematics are seldom directly applied; it is the definitions that are really useful. Once you see the definition of a differential equation, you see differential equations all over... If you want to apply mathematics, you have to live the life of differential equations. When you live this life, you can then go back to molecular biology with a new set of eyes that will see things that you could not otherwise see."

- Gian Carlo Rota

“The facts of mathematics are verified and presented by the axiomatic method. One must guard, however, against confusing the presentation of mathematics with the *content* of mathematics. An axiomatic presentation of a mathematical fact differs from the fact that is being presented as medicine differs from food... Confusing mathematics with the axiomatic method for presentation is as preposterous as confusing the music of Johann Sebastian Bach with the techniques for counterpoint in the Baroque age.”

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