Automating Mathematics?

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- We must have, if necessary invent, objective measures to see whether, and how well the tasks are performed.
- It may be useful to invent tasks as exercises.

Outline

1. Computer Assisted Mathematics

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2. Mathematical Tasks

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3. Artificial Inteligence elsewhere

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4. Mathematical Tasks revisited

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5. Conclusions

Computer Assisted Mathematics

What computers can do

- Numerical computation.

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- Enumeration.

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- Compact Enumeration.

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- The 290 Theorem for integral quadratic forms.

Robbins Conjecture: Deductive proofs

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- This is a Resolution Theorem Prover with Paramodulation.

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- In practice these have been used (so far) in formalizing proofs, not discovery.
- The greatest success so far has been the formal proof of the Feit-Thompson theorem by Georges Gonthier.

Homogeneous length functions on groups

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- $l(ghg^{-1}) = l(h)$ *for all* $g, h \in \langle \alpha, \beta \rangle$.
- $l(g^n) = nl(g)$ *for all* $g \in \langle \alpha, \beta \rangle$, $n \in \mathbb{Z}$.

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  - leading to the general feeling that $l([\alpha, \beta]) = 0$;
  - increasingly sharp bounds and methods of combining bounds, but no visible path to $l([\alpha, \beta]) = 0$.
- On Thursday morning I posted a proof of a computer-assisted bound.

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- $|\bar{b}\bar{a}b| \leq 1.0$ using $|\bar{a}| \leq 1.0$
- $|\bar{b}| \leq 1.0$
- $|a\bar{b}\bar{a}| \leq 1.0$ using $|\bar{b}| \leq 1.0$
- $|\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 2.0$ using $|\bar{a}\bar{b}a| \leq 1.0$ and $|b\bar{a}\bar{b}| \leq 1.0$
- ... (119 lines)
- $|ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 13.859649122807017$ using $|ab\bar{a}| \leq 1.0$ and $|\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 12.859649122807017$
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i.e., $l(\alpha, \beta) \leq 0.8152734778121775$

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Lemma

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- A probabilistic argument of Tao showed $l([x,y]) = 0$.

Mathematical Tasks

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- (Usually) depending on contexts and goals.

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- Find good and useful proofs, in particular proofs from which we can learn.

Artificial Inteligence elsewhere

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- Using this, we recursively decide the best moves based on alternately maximizing and minimizing.

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- Deep blue (which defeated Kasparov in 1997) and other top chess engines are such systems.

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- In a different domain, these weaknesses may matter much more than in Chess.

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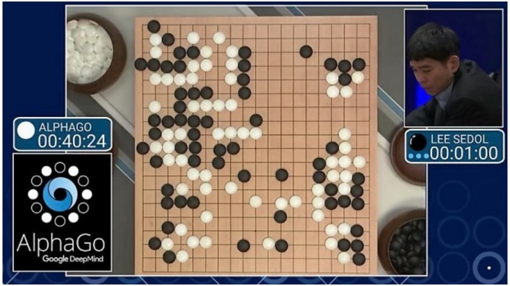

March 2016 vs Lee Sedol
May 2017 vs Ke Jie

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- We can optimize functions within this class using a gradient flow layer-by-layer.

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- AlphaGo came up with unexpected moves.

Word2Vec : Representation learning

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- The vectors capture analogy relations:

$$king - man + woman \approx queen.$$

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AlphaGo Zero

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- AlphaZero played a bold positional game.

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- The system has an internal representation which seems to be based on meanings of sentences.

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- For example the discriminative network tries to distinguish between real images and synthetic ones generated by the generative network.

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Mathematical Tasks revisited

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- HoTT foundations gives reasonable policies, values.
- More structure than Chess, more depth than Go.

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- Find good proofs – from which we can learn.

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  - Search for objects with desired properties, combining various approaches.