Lengths on Free groups

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- This the story of the answer and its discovery.

PolyMath 14 Participants

- Tobias Fritz, MPI MIS
- Siddhartha Gadgil, IISc, Bangalore
- Apoorva Khare, IISc, Bangalore
- Pace Nielsen, BYU
- Lior Silberman, UBC
- Terence Tao, UCLA

Outline

1. The Question

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2. Some lengths

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3. The Quest
4. The Theorem and Proof

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5. Computer Bounds and Proofs

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6. Epilogue

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- A pseudo-length function l on a group G is said to be a length function if $l(g) > 0$ for all $g \in G \setminus \{e\}$.
- Norms on vector spaces, such as $l(x, y) = \sqrt{x^2 + y^2}$ on $\mathbb{R}^2$, are length functions.

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- A pseudo-length function l on a group G is said to be conjugacy invariant if $l(ghg^{-1}) = l(h)$ for all $g, h \in G$ – if G is abelian every pseudo-length function is conjugacy-invariant.

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- Conversely any left invariant metric gives a length $l(g) := d(e, g)$, with $d(x, y) = l(x^{-1}y)$.
- The metric d associated to l is right-invariant, (i.e., $d(xg, yg) = d(x, y)$ for all $g, x, y \in G$) if and only if l is conjugacy invariant.

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- $l(g^n) = nl(g)$ *for all* $g \in \langle \alpha, \beta \rangle$, $n \in \mathbb{Z}$.

Some lengths

Word length

- The *word length* $l_w(g)$ *of an element* $g \in \langle \alpha, \beta \rangle$ *is the number of letters in the unique reduced word representing* g.

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- The word length is not conjugacy invariant as $l_w(\alpha\beta\alpha^{-1}) = 3 \neq 1 = l(\beta)$.
- It is also not homogeneous as $l_w((\alpha\beta\alpha^{-1})^2) = l_w(\alpha\beta^2\alpha^{-1}) = 4 \neq 2l_w(\alpha\beta\alpha^{-1})$.

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- The length $l_{\mathbb{Z}^2}(x, y) = |x| + |y|$ on $\mathbb{Z}^2$ induces a homogeneous, conjugacy-invariant pseudo-length $\bar{l}(g) = l_{\mathbb{Z}^2}(ab(g))$ on $\langle \alpha, \beta \rangle$.
- However this is not a length as $ab(\alpha\beta\alpha^{-1}\beta^{-1}) = (0, 0)$, $\bar{l}(\alpha\beta\alpha^{-1}\beta^{-1}) = 0$.

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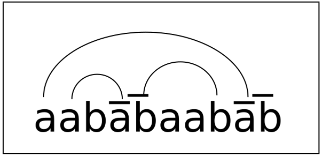
- Homogeneity and conjugacy-invariance are inherited by l_G from l_H.
- But l_G satisfies positivity if and only if $l_H|_{\phi(G)}$ satisfies positivity and φ is injective.

Non-crossing matchings

- Consider *non-crossing matchings* for a word in the letters α, β, α^{-1}, and β^{-1};



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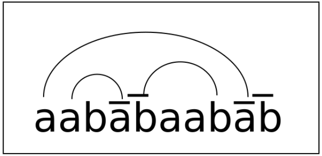

  - letters can only be matched with their inverses,
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- The *energy* is the number of unmatched letters.

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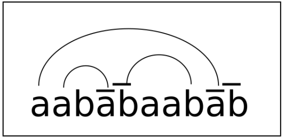
  - for all $g \in \langle \alpha, \beta \rangle$, $l(g) \leq l_{WC}(g)$.
- However l_{WC} is not homogeneous; if $g = \alpha[\alpha, \beta]$, then $l_{WC}(g) = 3$ but $l_{WC}(g^2) = 4$.



The Quest

Some observations

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- (Fritz) Homogeneity implies conjugacy invariant.
- (Tao, Khare) Homogeneity follows from $l(g^2) \geq 2l(g)$ for all $g \in \langle \alpha, \beta \rangle$.

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  - increasingly sharp bounds and methods of combining bounds were found, but there was no visible path to proving $l(\alpha\beta\alpha^{-1}\beta^{-1}) = 0$.
- On Thursday morning I posted a proof of a computer-assisted bound on $l(\alpha\beta\alpha^{-1}\beta^{-1})$.

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- $|\bar{b}\bar{a}b| \leq 1.0$ using $|\bar{a}| \leq 1.0$
- $|\bar{b}| \leq 1.0$
- $|a\bar{b}\bar{a}| \leq 1.0$ using $|\bar{b}| \leq 1.0$
- $|\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 2.0$ using $|\bar{a}\bar{b}a| \leq 1.0$ and $|b\bar{a}\bar{b}| \leq 1.0$
- ... (119 lines)
- $|ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 13.859649122807017$ using $|ab\bar{a}| \leq 1.0$ and $|\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 12.859649122807017$
- $|ab\bar{a}\bar{b}| \leq 0.8152734778121775$ using $|ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 13.859649122807017$ by taking 17th power.

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i.e., $l(\alpha, \beta) \leq 0.8152734778121775$

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Lemma

Let $f(m,k) = l(x^m[x,y]^k)$. *Then*

$$f(m,k) \leq \frac{f(m-1,k) + f(m+1,k-1)}{2}.$$

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$$f(m,k) \leq \frac{f(m-1,k) + f(m+1,k-1)}{2}.$$

- Using Probability, Tao showed $l(\alpha\beta\alpha^{-1}\beta^{-1}) = 0$.

The Theorem and Proof

The main results

Theorem

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Corollary

If G is not abelian (e.g. $G = \mathbb{F}_2$) there is no homogeneous length function on G.

Internal Repetition trick

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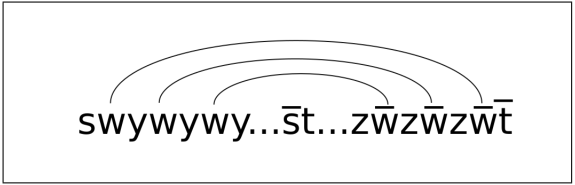

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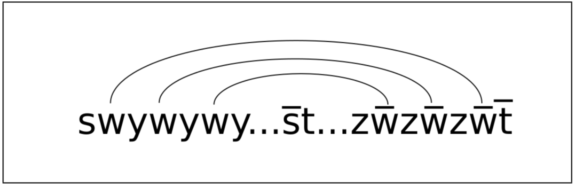

- Use $l(x) = \frac{l(x^n x^n)}{2n}$ and take limits.

The key inequality

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- Hence $l(x^m[x,y]^k) \leq \frac{l(x^{m-1}[x,y]^k)+l(y^{-1}x^m[x,y]^{k-1}xy)}{2}$.
- Since $y^{-1}x^m[x,y]^{k-1}xy$ is conjugate to $x^{m+1}[x,y]^{k-1}$, the claim follows.

Tao's probability theory argument

- The inequality $f(m, k) \leq \frac{f(m-1,k)+f(m+1,k-1)}{2}$ can be interpreted as the average of f being non-decreasing along the random walk on $\mathbb{Z}^2$ where we move by $(-1, 0)$ or $(1, -1)$ with equal probability.

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- This was bounded using the Chebyshev inequality.

Computer Bounds and Proofs

Bounds from Conjugacy invariance

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  - If $\xi_k = \xi_1^{-1}$, by the triangle inequality and conjugacy invariance
  $$l(g) \leq l(\xi_2\xi_3 \dots \xi_{k-1}) + l(\xi_{k+1}\xi_{k+2} \dots \xi_n)$$
  as $l(\xi_1\xi_2 \dots \xi_k) = l(\xi_1\xi_2 \dots \xi_{k-1}\xi_1^{-1}) = l(\xi_2\xi_2 \dots \xi_{k-1})$.

The recursive algorithm

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  - define $L(g) := \min(\{\lambda_0\} \cup \Lambda)$.
- In general, we may also have elements g for which $L(g)$ is given (or previously computed).

Ad hoc bounds using Homogeneity

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- Further, we can use this (in general improved) bound (as $L(g)$) recursively in the above algorithm.
- We computed such bounds in interactive sessions.
- The words used were $\alpha[\alpha, \beta]^k$, chosen based on non-homogeneity of the function l_{WC}.

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- In this case, we can instead view our algorithm as just keeping track of relevant inequalities.

Epilogue

Quasification

- The function $l : G \to [0, \infty)$ is a quasi-pseudo-length function if there exists $c \in \mathbb{R}$ such that $l(gh) \leq l(g) + l(h) + c$, for all $g, h \in G$.

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- We see that for a homogeneous quasi-pseudo-length function, $l([x, y]) \leq 4c$ for all $x, y \in G$.
- For a group with vanishing stable commutator length, e.g. $G = Sl(3, \mathbb{Z})$, any homogeneous quasi-pseudo-length function is bounded distance from a pullback from $G/[G, G]$.

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- Are there any general lessons for finding computer proofs, especially without expert knowledge?
- We can use the *families* $g_k = \alpha[\alpha, \beta]^k$, $k = 1, 2, 6$ and use $l(g_k) \leq \frac{l(g_k)^n}{n}$ with $n = 1, 2, \ldots, 20$.

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