Algebraic Topology - Jan-Apr. 2011, IISc.

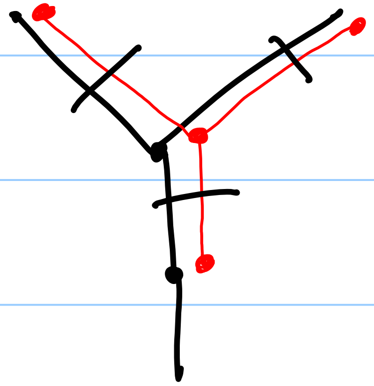
5/1/2011

- Topology ⟶ Algebra
- There is loss of information (cartoon not photo)
- In two steps:

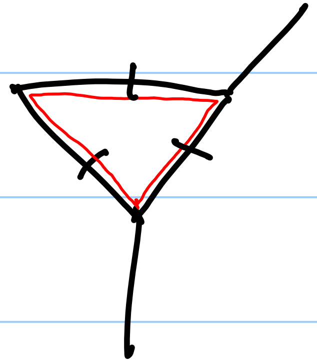
Spaces ⟶ Chain complexes ⟶ Homology,
Differential Graded Ring — Cohomology etc.

- Maps of spaces ⟶ Maps of Algebraic objects.

Loops in Graphs: [H_1 with $\mathbb{Z}/2\mathbb{Z}$ - coefficients]



A



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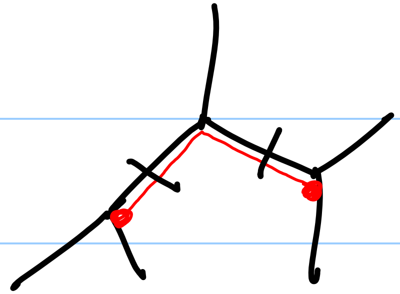
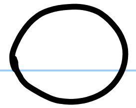


C

- Only B has a loop, in red
- This is a collection of edges with no boundary.

Loop: Collection of edges of a graph Γ whose boundary is trivial.



C_1 = set of collections of edges

= set of subsets of the set E of edges

= Vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis edges E.

Why are these the same?

V = vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $E = \{e_1, \dots, e_k\}$

$$= \{(a_1 e_1 + a_2 e_2 + \dots + a_k e_k) : a_j \in \{0,1\}\}$$

$$\updownarrow$$

$$\{e_j : a_j = 1\}$$

E.g. If $E = \{e_1, \dots, e_5\}$,

the element $e_1 + e_2 + e_5$ corresponds to the set $\{e_1, e_2, e_5\}$

$$e_1 + e_2 + e_5 = 1\cdot e_1 + 1\cdot e_2 + 0\cdot e_3 + 0\cdot e_4 + 1\cdot e_5$$

- C_1 is the vector space as defined above.

Ex

- Addition in C_1 corresponds, in terms of sets, to symmetric difference $A \triangle B = (A \cup B) \setminus (A \cap B)$.

E.g.
$$\overset{\{e_1, e_3\}}{\overset{\updownarrow}{(e_1 + e_3)}} + \overset{\{e_2, e_3)}{\overset{\updownarrow}{(e_2 + e_3)}} = e_1 + e_2 + 0\cdot e_3 \leftrightarrow \{e_1, e_2\}$$

Thus, the vector space C_1 captures subsets of the set of edges.

C_0 = Set of subsets of vertices V

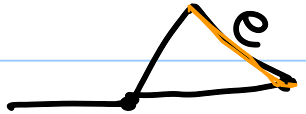
= Vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis V.

Boundary:

$\partial_1 = \partial : C_1 \longrightarrow C_0$, homomorphism (linear map)

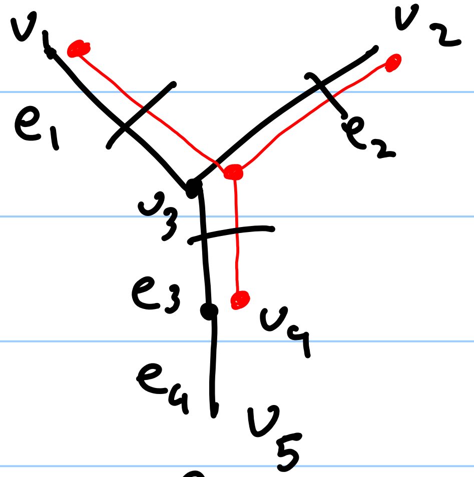
- We specify ∂ on the basis of C_1, namely E

$\partial e = v_0 + v_1$ where v_0 & v_1 are the endpoints (vertices) of e.



Loop: Element z_1 of C_1 with $\partial z_1 = 0$.

Examples: $z_A \in C_1(A)$, $z_B \in C_1(B)$, $z_C \in C_1(C)$



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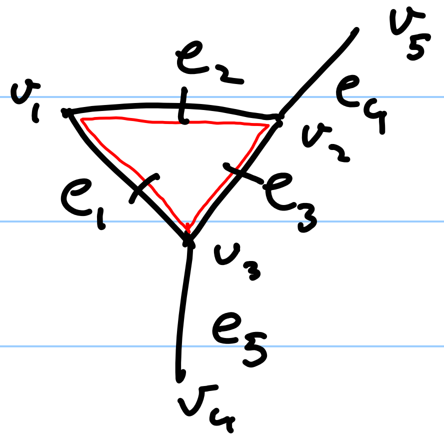

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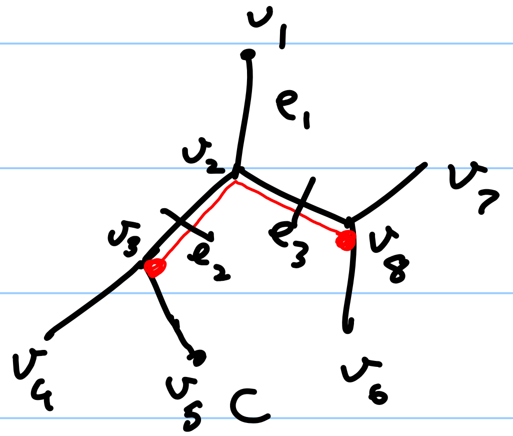


C

$z_A = e_1 + e_2 + e_3$; $\partial z_A = v_1 + v_2 + v_3 + v_4$

$z_B = e_1 + e_2 + e_3$; $\partial z_B = \partial e_1 + \partial e_2 + \partial e_3 = \begin{cases} (v_1+v_3) + (v_1+v_2) \\ + (v_2+v_3) = 0 \end{cases}$

$z_C = e_2 + e_3$; $\partial z_C = v_3 + v_2 + v_2 + v_8 = v_3 + v_8$

__Exercise:__ The coefficient of v in ∂z is 1 iff v is contained in an odd number of edges with coefficient 1 in z.

Thus, $\partial z_A \neq 0$, $\partial z_B = 0$, $\partial z_C \neq 0$, which is consistent with our intution that z_B is a loop.

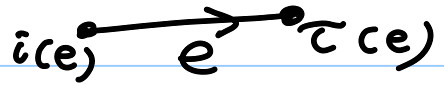
Homology:

Defn: $H_1 = \ker(\partial : C_1 \longrightarrow C_0)$

= 'set of loops' (Actually vector space with basis loops)

Paths and Loops in graphs.

- Let Γ be a graph, V = vertices, E = edges
- For an oriented edge e, we can define its initial vertex $i(e)$ and terminal vertex $\tau(e)$



- A path in Γ is a sequence of oriented edges $e_1, \dots, e_n$ s.t. $\tau(e_1) = i(e_2)$, $\tau(e_2) = i(e_3) \dots$
- A path is said to be a loop if

$$\tau(e_n) = i(e_1)$$

Relation between paths and C_* (i.e., C_0, C_1)

- A path $e_1, \dots, e_n$ gives an element of C_1, namely $e_1 + e_2 + \dots + e_n$ (called a 1-chain)

Proposition: $\partial(e_1 + \dots + e_n) = i(e_1) + \tau(e_n)$

Proof: $\partial(e_1 + \dots + e_n) = \partial e_1 + \partial e_2 + \dots + \partial e_n$

$$= (i(e_1) + \tau(e_1)) + (i(e_2) + \tau(e_2)) + \dots + (i(e_n) + \tau(e_n))$$

$$= i(e_1) + \underset{\overset{\shortparallel}{0}}{(\tau(e_1) + i(e_2))} + \dots + \tau(e_n)$$

$$= i(e_1) + \tau(e_n)$$

Corollary: The path $e_1, \dots, e_n$ is a loop iff the corresponding 1-chain

$$z = e_1 + \dots + e_n$$

satisfies $\partial z = 0$

Defn: A chain z with $\partial z = 0$ is called a cycle.

- These are 'loops' in the graph.
- $H_1 =$ The vector space of cycles.

Connectedness:

Defn: The graph Γ is connected if given vertices v_1 & v_2 in Γ, there is a path $e_1, \dots, e_n$ with $\iota(e_1) = v_1$ and $\tau(e_n) = v_2$ [path from v_1 to v_2].

Excercise: Show that this is the same as connected in the topological sense.

- A pair of vertices v_1, v_2 gives $v_1 + v_2 \in C_0$
- A path $e_1, \dots, e_n$ gives $\xi_1 = e_1 + \dots + e_n \in C_1$
- $e_1, \dots, e_n$ is a path from v_1 to v_2 iff $\partial\xi = v_1 + v_2$

Theorem: Given vertices v_1 and v_2, there is a path $\gamma = e_1, \ldots, e_n$ from v_1 to v_2 iff $\exists\, \xi \in C_1$ s.t. $\partial \xi = v_1 + v_2$

- We say $v_1 + v_2$ is a boundary.

10/1/2011

Lecture 2:

- Γ is a graph
- We associate to Γ

$$C_1(\Gamma) \xrightarrow{\partial_1} C_0(\Gamma) \quad \text{(chain complex)}$$

with C_1 and C_0 vector spaces over $\mathbb{Z}/2\mathbb{Z}$

- A path $\gamma = e_1, ---, e_n$ has associated to it an element

$$\gamma_\# = e_1 + \ldots + e_n \in C_1(\Gamma)$$

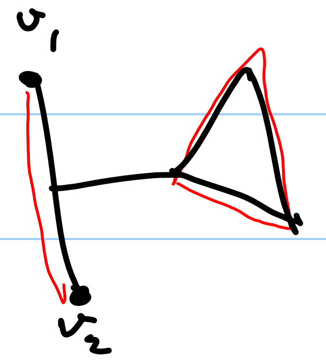
- $\partial \gamma_\# = i(e_1) + \tau(e_n)$

Theorem: There is a path γ from v_1 to v_2 iff $\exists z \in C_1$ s.t. $\partial z = v_1 + v_2$ (i.e., $v_1 + v_2$ is a boundary).

Pf: · If γ is a path from v_1 to v_2, then

$$\partial \gamma_\# = v_1 + v_2$$

· Coversely, let $\partial z = v_1 + v_2$



· We prove by induction on the number of terms in z.

· If $z = e$ has a single term, e is a path from v_1 to v_2 (as $\partial e = v_1 + v_2$)

Otherwise, as $\partial z = v_1 + v_2$, there is an edge e_1 with coefficient 1 in z s.t. $\partial e_1 = v_1 + v_1'$, $v_1' \neq v_1$.

- Now $z' = z + e_1$ has fewer terms than z, and $\partial z' = \partial z + \partial e_1 = v_1 + v_2 + v_1 + v_1' = v_1' + v_2$
- By induction hypothesis $\exists$ path $e_2, \dots, e_n$ from v_1' to v_2
- $\gamma = e_1, \dots, e_n$ is a path from v_1 to v_2.

Remark: · We can assume γ has no edge repeated.

- We can also assume every edge in γ is in $\{e_1, \dots, e_n\}$ if $z = e_1 + \dots + e_n$.

Cor: Γ is connected iff $\forall v_1, v_2 \in \overset{\text{vertex set}}{\overset{\shortparallel}{V}}(\Gamma)$,

$v_1 + v_2 \in \operatorname{im}(\partial_1)$

Definition: $H_0(\Gamma) = C_0(\Gamma) / \operatorname{im}(\partial_1)$

Augmentation homorphism: $\varepsilon : C_0(\Gamma) \longrightarrow \mathbb{Z}/2\mathbb{Z}$

$$\varepsilon(v) = 1 \quad \forall v \in V(\Gamma)$$

- In general $\varepsilon(v_1 + \dots + v_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

- $\varepsilon(v_1 + v_2) = 0$

Lemma: $\ker(\varepsilon) = \mathrm{Span}\{v_1 + v_2 : v_1, v_2 \in V(\Gamma)\}$

Pf: $z = v_1 + \dots + v_n \in \ker(\varepsilon) \iff n = 2m$ is even

$$\Rightarrow v = (v_1 + v_2) + (v_3 + v_4) + \dots \in \mathrm{Span}\{v_1 + v_2; \dots\}$$

Conversely $v_1 + v_2 \in \ker(\varepsilon)$ $\forall v_1, v_2 \Rightarrow \mathrm{Span}\{\dots\} \subset \ker(\varepsilon)$

Thus,

Cor: Γ is connected iff $\mathrm{im}(\partial_1) = \ker(\varepsilon)$

· ε is onto if Γ is non-empty,

hence $H_0(\Gamma) = C_0(\Gamma)/\mathrm{im}(\partial_1) = C_0/\ker(\varepsilon) = \mathbb{Z}/2\mathbb{Z}$

Theorem: Γ is connected iff

$$H_0(\Gamma) = \mathbb{Z}/2\mathbb{Z}.$$

Loops and H_1:

- Γ a graph
- Chain complex: $C_1(\Gamma) \xrightarrow{\partial_1} C_0(\Gamma)$
- $H_1(\Gamma) = \ker(\partial_1)$ $\quad \left[H_0(\Gamma) = \underset{\underset{\text{coker}(\partial_1)}{\shortparallel}}{C_0(\Gamma)/\text{im}(\partial_1)} \right]$

Theorem: Γ has a non-trivial loop
(i.e. path with initial point = final point)
γ with no repeated edges iff $H_1(\Gamma) \neq 0$.

Pf: If $\gamma = e_1, \ldots, e_n$ is a loop with no repeated edges, then $\cdot\ \gamma_\# = \overset{\in C_1}{e_1 + \ldots + e_n} \neq 0$

and $\cdot\ \partial_1 \gamma_\# = 0$

Thus $0 \neq \gamma_\# \in \ker(\partial_1) = H_1(\Gamma) \Rightarrow H_1(\Gamma) \neq 0$.

Conversely,

suppose $z = e_1 + \dots + e_n \in H_1(\Gamma)$, $z \neq 0$.

let $\partial_1 e_1 = v_0 + v_1$, $z' = z + e_1 = e_2 + \dots + e_n$.

- By our previous theorem, as

$$\partial_1 z' = \partial_1 z + \partial_1 e_1 = v_0 + v_1,$$

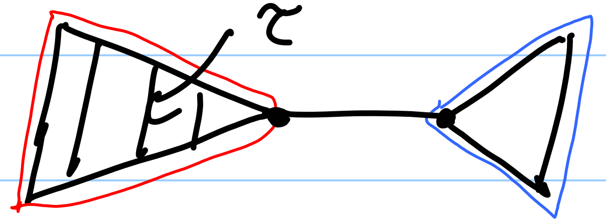
there is a path γ from v_1 to v_0, which we can see has

  - No repeated edges
  - Edges in $\{e_2, \dots, e_n\}$

- Thus, e_1 followed by γ is a path with no repeated edges.

Introducing Δ^{les}:

E.g.



- Suppose we fill in some Δ^{les} (2-simplices) in Γ to get a space X
- We can define C_2 = vector space$/\mathbb{Z}/2\mathbb{Z}$ with basis Δ^{les}
- $\partial_2 : C_2 \longrightarrow C_1$ with $\partial_2 \tau = e_1 + e_2 + e_3$, e_i edges of τ.

We thus obtain $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$: chain complex

$$H_1(X) = \ker(\partial_1) / \text{im}(\partial_2)$$

12/1/2011

Lecture 3:

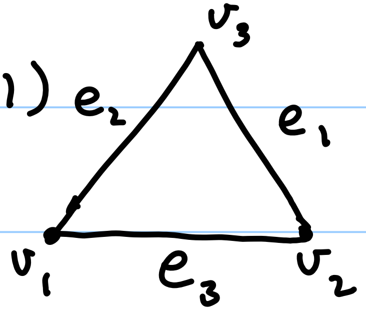
- Γ graph gives $C_1(\Gamma) \xrightarrow{\partial_1} C_0(\Gamma)$,

  $C_1(\Gamma) =$ Vector space with basis $E(\Gamma)$ (over $\mathbb{Z}/2\mathbb{Z}$)

  $C_0(\Gamma) =$ Vector space with basis $V(\Gamma)$ $\overset{\wr}{F_2}$

- $H_1(\Gamma) = \ker(\partial_1)$

E.g. (1)



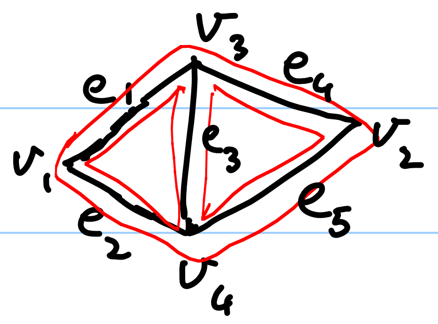
$C_1 = \{(a_1e_1 + a_2e_2 + a_3e_3) : a_1, a_2, a_3 \in F_2\}$

$\partial_1(a_1e_1 + \underset{\overset{\shortparallel}{z}}{a_2}e_2 + a_3e_3) = (a_2+a_3)v_1 + (a_1+a_3)v_2 + (a_1+a_2)v$,

$\therefore\ \partial_1 z = 0 \Longleftrightarrow a_2 + a_3 = a_1 + a_3 = a_1 + a_2 = 0$

$\Longleftrightarrow a_1 = a_2 = a_3 = a$ (say)

$\therefore\ H_1(\Gamma) = \ker(\partial_1) = \{a(e_1 + e_2 + e_3) : a \in F_2\} \cong F_2$

(2) 

Rk: · $e_1+e_2+e_3$, $e_3+e_4+e_5$ and

$e_1+e_2+e_4+e_5$ are all cycles

i.e. $\partial_1(\cdot)=0$

· $e_1+e_2+e_4+e_5=(e_1+e_2+e_3)+(e_3+e_4+e_5)$

· $C_1 \xrightarrow{\partial_1} C_0$

Let $z = a_1e_1+a_2e_2+a_3e_3+a_4e_4+a_5e_5$

$$\partial_1 z = (a_1+a_2)v_1+(a_4+a_5)v_2+(a_1+a_3+a_4)v_3+(a_2+a_3+a_5)v_4$$

$$\cdot\ \partial_1 z = 0 \Leftrightarrow \begin{cases} a_1+a_2=0 \Leftrightarrow a_1=a_2 & \cdots(1) \\ a_4+a_5=0 \Leftrightarrow a_4=a_5 & \cdots(2) \\ a_1+a_3+a_4=0 \Leftrightarrow a_3=a_1+a_4 & \cdots(3) \\ a_2+a_3+a_5=0 \Leftrightarrow a_3=a_2+a_5 \Leftrightarrow (3) & \text{given } (1)\&(2) \end{cases}$$

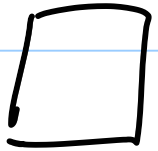
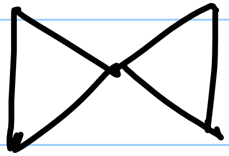
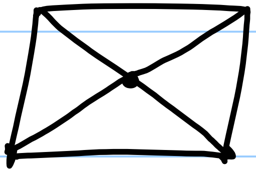
$$\therefore\ \partial_1 z = 0 \Leftrightarrow z = a_1e_1+a_1e_2+(a_1+a_4)e_3+a_4e_4+a_4e_5$$

$$\therefore\ H_1(\Gamma)=\{a_1e_1+a_1e_2+(a_1+a_4)e_3+a_4e_4+a_4e_5 : a_1,a_4\in F_2\}$$

$$\simeq F_2^2=\{(a_1,a_4): a_1,a_4\in F_2\}$$

i.e. H_0, H_1

Exercises: Compute the homology of :

(a)  (b)  , (c) 

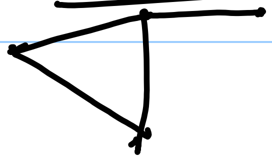
edge with $i(e)=\tau(e)$

On to higher dimensions:

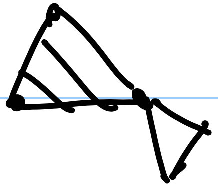
Motivation: A graph Γ without loops or multiple edges is determined by

Not allowed

Allowed



- Set of vertices V
- Pairs of vertices that bound an edge.



- We further specify triples $\{v_0, v_1, v_2\}$ that bound a Δ^{le} $\Rightarrow$ $\{v_0, v_1\}$ etc. bound edges

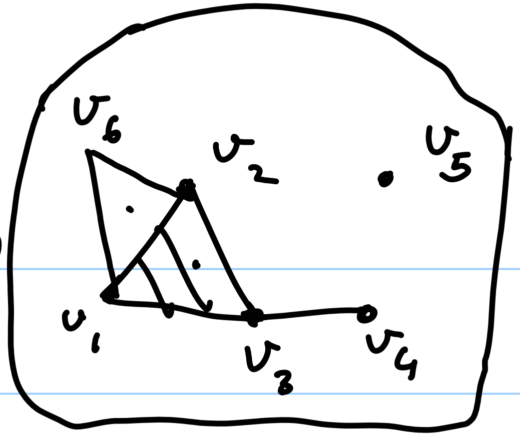
Simplicial Complex:

A simplicial complex Σ is a pair $(V(\Sigma), S(\Sigma))$, $S(\Sigma) \subset \mathcal{P}(V(\Sigma))$ is a collection of non-empty subsets $V(\Sigma)$ s.t.

- if $v \in V(\Sigma)$, $\{v\} \in S(\Sigma)$.
- if $\sigma \in S(\Sigma)$, $\tau \subset \sigma$, $\tau \neq \emptyset \Rightarrow \tau \in S(\Sigma)$

- An element $\{v_0, \dots, v_n\}$ of $S(\Sigma)$ is called an n-simplex.

E.g. :(1)



$= \quad \Sigma$

$$V(\Sigma) = \{v_1, v_2, v_3, v_4, v_5\}$$

$$S(\Sigma) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\},$$
$$\{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_6\}, \{v_2, v_6\}\}$$

E.g.(2) Poset complex : Let $(X, \leq)$ be a partially ordered set. We associate to X a simplicial complex

$$V(\Sigma) = X, \ S(\Sigma) = \begin{cases} \text{non-empty subsets of } X \text{ that} \\ \text{are totally ordered} \end{cases}$$

Exercise: Show that this is a simplicial complex.

Example: (3) G a finite group, p a prime

- The set of proper (non-empty) subgroups of G of order p^n for some n is a poset.
- We can associate to this a simplicial complex.

Conjecture: G contains a normal p-group iff this simplicial complex is contractible.

(4) If $\Sigma = (V(\Sigma), S(\Sigma))$ is a simplicial complex, then $S(\Sigma)$ is a poset, which hence gives a simplicial complex. This is called the barycentric subdivision.

Geometric realisation of a simplicial complex

- $\Sigma = (V(\Sigma), S(\Sigma))$, we define
- $|\Sigma| = \{a_0 v_0 + \dots + a_n v_n : v_0, \dots, v_n \in V(\Sigma), \{v_0, \dots, v_n\} \in S(\Sigma)$
  $: a_i \in \mathbb{R}, a_i \geq 0, \sum_{i=0}^{n} a_i = 1\}$

i.e., points of $|\Sigma|$ are convex linear combinations of vertices of Σ that bound a simplex

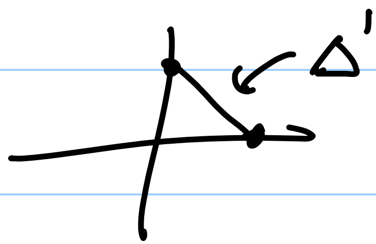
- We will define a topology on this, in terms of simplices.

Simplices.

- The standard n-simplex is

$$\Delta^n = \{ (a_0, a_1, \dots, a_n) : a_i \geq 0, \sum_{i=0}^{n} a_i = 1 \} \subseteq \mathbb{R}^{n+1}$$

e.g. $\Delta^1 =$



(with the subspace topology)

More generally, if $v_0, \dots, v_n \in \mathbb{R}^N$ are points such that $\{v_1 - v_0, v_2 - v_0, \dots, v_n - v_0\}$ are independent vectors, then they span a simplex

$$\langle v_0, \dots, v_n \rangle = \{ \sum a_i v_i : a_i \geq 0, \sum a_i = 1 \}.$$

17/1/2011

Lecture 4

- Standard n-simplex:

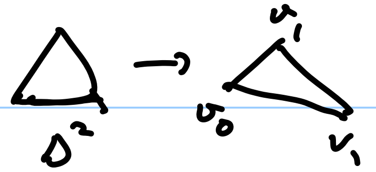
$$\Delta^n = \{(a_0, \dots, a_n) : a_i \geq 0, \sum_{i=0}^{n} a_i = 1\}$$

$$= \text{Convex hull of unit vectors } (0, \dots, 1, \dots, 0) \in \mathbb{R}^{n+1}$$

- Given $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^N$ s.t. $\{v_i - v_0\}_{i=1}^{n}$ is independent the convex hull of $v_0, \dots, v_n$ is called the n-simplex with vertices $v_0, \dots, v_n$ (denoted $\langle v_0, \dots, v_n \rangle$)

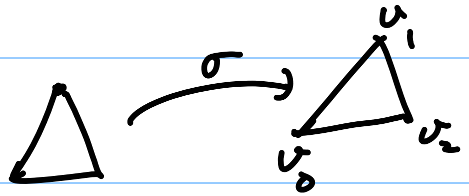
- Canonical map: $\sigma : \Delta^n \longrightarrow \langle v_0, \dots, v_n \rangle$ is

$$\sigma : (a_0, \dots, a_n) \longmapsto \sum_{i=0}^{n} a_i v_i$$



Exercise: Show this is a homeomorphism.

- 'Orientations': The canonical map from Δ^n to the convex hull of $\{v_0, \dots, v_n\}$ is determined once an order on $\{v_0, \dots, v_n\}$ is fixed.



- Such an order is called an 'orientation' of the simplex $\langle v_0, \dots, v_n \rangle$.
- The standard n-simplex has an obvious orientation. (vertices are $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$ ....

Recall:

Simplicial Complex:

A simplicial complex Σ is a pair $(V(\Sigma), S(\Sigma))$, $S(\Sigma) \subset \mathcal{P}(V(\Sigma))$ is a collection of non-empty finite subsets $V(\Sigma)$ s.t.

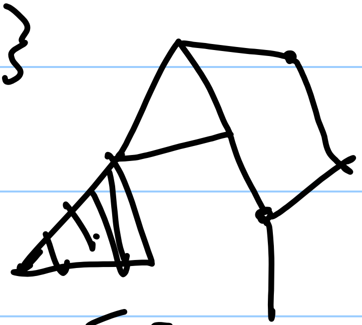
- if $v \in V(\Sigma)$, $\{v\} \in S(\Sigma)$.
- if $\sigma \in S(\Sigma)$, $\tau \subset \sigma$, $\tau \neq \emptyset \Rightarrow \tau \in S(\Sigma)$

- An element $\{v_0, \ldots, v_n\}$ of $S(\Sigma)$ is called an n-simplex.

Geometric Realisation: Σ a simplicial complex

$$|\Sigma| = \Big\{ \underbrace{\sum_{i=0}^{k} a_i v_i}_{\text{(formal linear combination)}} : a_i \geq 0, \sum_{i=0}^{k} a_i = 1, v_i \in V(\Sigma), \{v_0, \dots, v_k\} \in S(\Sigma) \Big\}$$

$\Sigma =$



· Here, $\sum_{i=0}^{k} a_i v_i$ is a 'formal linear combination' of vertices, i.e., just number $a_0, \dots, a_k$ associated to the vertices (v_i - dummy letters)

'Formal linear combinations': Given a set S, a formal linear combination of elements in S is an 'expression' $\sum_{i=1}^{k} a_i s_i$ where $s_i \in S$ are elements.

Formal view:

- Given a set S, we consider functions

$$\alpha : S \longrightarrow \mathbb{R}$$

with the property that the set

$$\{s \in S : \alpha(s) \neq 0\}$$ is finite.

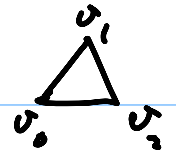
- A function as above can be identified with the expression: $\sum\limits_{s \in S} \alpha(s) \cdot s$.

- In $|\Sigma|$, '$\sum\limits_{i=0}^{n} a_i v_i$' is a function $\alpha : V(\Sigma) \to \mathbb{R}$ with $\alpha(v_i) = a_i$ & $\alpha(v) = 0$ if $v \notin \{v_0 \dots, v_n\}$.

'Orientation': An orientation on Σ is a total order on each simplex $\{v_0, \dots, v_n\} \in S(\Sigma)$ s.t. the restriction to each subset of $\{v_0, \dots, v_n>$ gives the total order on this subset.

<u>E.g.</u> We can take a total order on $V(\Sigma)$ and restrict to all simplices.

- $\langle v_0, \dots, v_k \rangle \in S(\Sigma)$ will denote a simplex $\{v_0, \dots, v_k\} \in S(\Sigma)$ with $v_0 < v_1 < \dots < v_k$.



Canonical maps & topology:

- Given $\alpha = \langle v_0, \dots, v_k \rangle \in S(\Sigma)$, we have a canonical map

$$\sigma_\alpha : \Delta^k \longrightarrow |\Sigma|$$

given by

$$\sigma_\alpha : (a_0, \dots, a_k) \longmapsto \sum_{i=0}^{k} a_i v_i \in |\Sigma|$$

(1) σ_α is 1-1 and $|\Sigma| = \bigcup_{\alpha \in S(\Sigma)} \sigma_\alpha(\Delta^k)$. (Excercise)

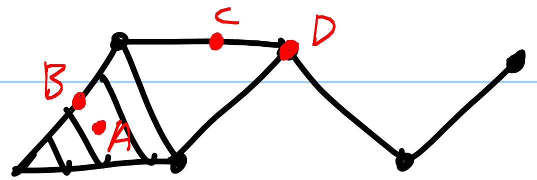
(2) if $\overset{k\text{-simplex}}{\alpha} \neq \overset{l\text{-simplex}}{\beta}$, $\alpha, \beta \in S(\Sigma)$, then

$$\sigma_\alpha(\mathring{\Delta}^k) \cap \sigma_\beta(\mathring{\Delta}^l) = \emptyset$$

(3) $|\Sigma| = \bigcup_{\alpha \in S(n)} \sigma_\alpha(\mathring{\Delta}^k)$ (α k-simplex)

Thus, (Convention: interior of $\Delta^0 = \Delta^0$).

- We have canonical maps $\sigma_\alpha : \Delta^k \to |\Sigma|$
- Each point in $|\Sigma|$ is contained in the interior of a unique simplex (more precisely, the image of the interior of a standard simplex under a unique canonical map)



- Topology on $|\Sigma|$: $U \subset |\Sigma|$ is open iff $\forall \alpha \in S(\Sigma)$, $\sigma_\alpha^{-1}(U)$ is open.

Simplicial Chain complexes:

- Let Σ be a simplicial complex (given orientation)
- Definition: $C_k(\Sigma)$ is the free abelian group with basis the k-simplices of Σ.

$$= \{\underset{\substack{\uparrow \\ \text{formal linear combination}}}{\sum_k a_k \sigma_k} : \sigma_k \in S(\Sigma) \text{ is a } k\text{-simplex}, a_k \in \mathbb{Z}\}$$

Digression now: R-modules, free R-modules etc.

Some Algebra:

Basic Example: Let V be a vector space over a field k.

Basis: $B = \{v_1, \dots, v_n\}$ form a basis of V if:

(1) Every $v \in V$ can be uniquely written as $\sum_{i=1}^{n} a_i v_i$, i.e., V can be identified with (formal) linear combinations of $\{v_1, \dots v_n\}$

(2) (Free) Given a function $l : B \to W$, W a vector spa., $\exists !$ $L : V \longrightarrow W$ linear transformation such that $L(v_i) = l(v_i)$ $\forall v_i \in B$.

Free abelian groups:

- Given $f: \{1\} \longrightarrow A$, A an abelian group,

$\exists!$ homomorphism $\varphi: \mathbb{Z} \longrightarrow A$ s.t. $\varphi(1) = f(1)$,

namely $\varphi(n) = n \cdot f(1)$.

- Given $f: \overset{B}{\overset{\shortparallel}{\{(1,0), (0,1)\}}} \longrightarrow A$, $\exists! \varphi: \mathbb{Z}^2 \longrightarrow A$

s.t. $\varphi(b) = f(b)$ $\forall b \in B$, namely $\boxed{\varphi(n,m) = n f((1,0)) + m f((0,1))}$

Definition: An abelian group A is a free abelian group with basis $B \subset A$ if given any function

$f: B \longrightarrow A$, A an abelian group,

$\exists! \varphi: A \to A$ homomorphism s.t. $\varphi(b) = f(b)$ $\forall b \in B$.

More examples:

- $\{2\}$ is not a basis of $\mathbb{Z}$, as given

  $f : \{2\} \longrightarrow \mathbb{Z}$, $f(2) = 1$

  $\nexists$ homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ with $\varphi(2) = f(2)$.

19/1/2011

Lecture 5:

Recall: An abelian group $\mathcal{A}$ is a free abelian group with basis $\mathcal{B}$, $\mathcal{B} \in \mathcal{A}$, if for all abelian groups A and functions $f: \mathcal{B} \to A$, there is a unique homomorphism $\varphi: \mathcal{A} \to A$ s.t.

$$\varphi(b) = f(b) \quad \forall\, b \in \mathcal{B}$$

- $\mathbb{Z}$ is a free abelian group with $\{1\}$ as a basis but $\{2\}$ is not a basis
- Every vector space has a basis.
- However, the analogue is not true for abelian groups.

Proposition: $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$, $n \geqslant 2$, is not free with any basis

Pf: Suppose $\mathcal{B} \subset \mathcal{A}$ is a basis.

- We cannot have $\mathcal{B} = \emptyset$ as given

$$f : \mathcal{B} \xrightarrow{=\emptyset} A = \mathcal{A} = \mathbb{Z}/n\mathbb{Z} \text{ (the unique function)}$$

there are two homomorphisms

$$\varphi_1, \varphi_2 : \mathcal{A} \to A, \text{ namely } \varphi_1 = 0 \ \& \ \varphi_2 = id$$

with $\varphi_1(b) = f(b) \ \forall b \in \mathcal{B}$ & $\varphi_2(b) = f(b) \ \forall b \in \mathcal{B}$.

- $\varphi_1 \neq \varphi_2$ as $n \geqslant 2$, hence $\varphi_1(1) \neq \varphi_2(1)$.

This contradicts uniqueness in the definition.

Exercise: $\emptyset$ is a basis for the group $\mathcal{A} = \{0\}$.

Thus, $B \neq \emptyset$, say $k \in B$, $k \in \mathbb{Z}/_{n\mathbb{Z}}$.

Let $A = \mathbb{Z}$, $f : B \to \mathbb{Z}$ be a function s.t.

$$f(k) = 1$$

Then there is no homomorphism $\varphi : A \to \mathbb{Z}$ s.t.

$$\varphi(k) = f(k) = 1$$

as this implies

$$0 = \varphi(0) = \varphi(nk) = n \cdot \varphi(k) = n \cdot f(k) = n$$

a contradiction.

Exercise (*): If A has an element of finite order, then A is not free.

Theorem: Every subgroup of a free abelian group is free. (Pf: Look at Algebra books)

- Torsion $\hookrightarrow$ Elements of finite order.

Proposition: $\mathbb{Q}$ is not a free abelian group.

Pf: Suppose $B \subset \mathbb{Q}$ is a basis, as before $B \neq \emptyset$.

Let $x \in B$. Take $A = \mathbb{Z}$,

$$f : B \to \mathbb{Z} \text{ s.t. } f(x) = 1$$

If $\varphi : \mathbb{Q} \to \mathbb{Z}$ is a homomorphism

$$1 = f(x) = \varphi(x) = 2 \cdot \varphi(x/2) \in 2\mathbb{Z}$$

a contradiction.

Theorem: Any finitely generated abelian group A without elements of finite order is free. In general, a finitely generated abelian group A is isomorphic to a group of the form

$$A = \mathbb{Z}^r \times \mathbb{Z}/_{n_1\mathbb{Z}} \times \mathbb{Z}/_{n_2\mathbb{Z}} \times \dots \times \mathbb{Z}/_{n_k\mathbb{Z}}.$$

Free abelian group with given basis $\mathcal{B}$:

Let $\mathcal{B}$ be a set.

- Consider $\mathcal{A} = \{ \alpha : \mathcal{B} \to \mathbb{Z} \text{ fn.} : \{\alpha(b) \neq 0\} \text{ is finite} \}$
- This is an abelian group. Consider the functions

$$\delta_b : \mathcal{B} \to \mathbb{Z}, \quad \delta_b(b') = \begin{cases} 1 & \text{if } b = b' \\ 0 & \text{otherwise} \end{cases}$$

Proposition: $\tilde{\mathcal{B}} = \{\delta_b\}_{b \in \mathcal{B}}$ is a basis of $\mathcal{A}$.

Pf: Given A an abelian group and $f : \tilde{\mathcal{B}} \to A$, the unique homomorphism $\varphi : \mathcal{A} \to A$ that agrees with f is $\varphi(\alpha) = \sum_{b \in \mathcal{B}} \alpha(b) \cdot f(\delta_b)$.

(use $\alpha = \sum_{b \in \mathcal{B}} \alpha(b) \cdot \delta_b$) $\square$

Thus, identifying $\mathcal{B}$ with $\tilde{\mathcal{B}}$, $b \leftrightarrow \delta_b$, we can regard $\mathcal{A}$ as a free abelian group with basis $\mathcal{B}$.

- Elements of $\mathcal{A}$ are

$$\begin{aligned} \mathcal{A} &= \{ \alpha : \mathcal{B} \to \mathbb{Z} : \alpha \text{ has finite support} \} \\ &= \{ \sum_{b \in \mathcal{B}} \alpha(b) \cdot \delta_b : \alpha \text{ has finite support} \} \\ &= \{ \sum_{b \in \mathcal{B}} \alpha_b \cdot b : \delta_b \leftrightarrow b, \ \alpha(b) = \alpha_b \} \end{aligned}$$

Thus, $\mathcal{A}$ is the set of formal linear combinations of elements in $\mathcal{B}$. Every element in $\mathcal{A}$ can be uniquely written as a linear combination in $\mathcal{B}$.

We shall see next.

(1) If $\mathcal{B}$ is a basis of $\mathcal{A}$, then every element in $\mathcal{A}$ can be uniquely written as a linear combination of elements in $\mathcal{B}$.

(2) If $\mathcal{B}$ & $\mathcal{B}'$ are bases of $\mathcal{A}$ & $\mathcal{A}'$, and $f: \mathcal{B} \to \mathcal{B}'$ is a bijection, then $\exists! \varphi: \mathcal{A} \to \mathcal{A}'$ isomorphism that agrees with f.

24/1/11

Lecture 6: Recall:

Free Abelian Groups (Universal property)

- $\mathcal{A}$ is free with basis $\mathcal{B}$ if given

$f: \mathcal{B} \longrightarrow A$ function, $\exists ! \, \varphi: \mathcal{A} \rightarrow A$ homomorphism

s.t.

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "f"] \arrow[dr, hook, "\text{inclusion}"'] & & A \\
& \mathcal{A} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

commutes (i.e., following arrows in different ways gives the same result)

Constructing free Abelian groups:

Given a set $\mathcal{B}$, we constructed a free abelian group, we shall call $\mathbb{Z}[\mathcal{B}]$, with $\mathcal{B}$ as a basis.

$$\mathbb{Z}[\mathcal{B}] = \{\alpha : \mathcal{B} \longrightarrow \mathbb{Z} \text{ fn.} : \alpha(b) = 0 \text{ for all but finitely many } b \in \mathcal{B}\}$$

$$= \Big\{\sum_{b\in\mathcal{B}} \alpha_b \cdot \underset{\substack{\updownarrow \\ b\in\mathcal{B}}}{\delta_b} : \alpha_b \in \mathbb{Z},\ \alpha_b = 0 \text{ for all but finitely...}\Big\}$$

$$= \Big\{\sum_{b\in\mathcal{B}} \alpha_b \cdot b : \alpha_b \in \mathbb{Z} \text{ finitely supported}\Big\}$$

Under this identification, $\sum_{b\in\mathcal{B}} \alpha_b \cdot b + \sum_{b\in\mathcal{B}} \alpha_b' \cdot b = \sum_{b\in\mathcal{B}} (\alpha_b + \alpha_b') \cdot b$

i.e. $\mathbb{Z}[\mathcal{B}]$ = 'formal linear combinations'

In particular,

- Every element in $\mathbb{Z}[B]$ is a unique linear combination of elements in B.

<u>Uniqueness for given basis</u>

- Let $f: B \longrightarrow B'$ be a bijection of sets, and A and A' be free abelian groups with bases B & B'.
- Hence $\exists \varphi: A \longrightarrow A'$ homomorphism s.t.

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, hook, "j"'] \arrow[dr, "j' \circ f"] & B' \arrow[d, hook, "j'"] \\
A \arrow[r, "\varphi"'] & A'
\end{tikzcd}$$

commutes

Theorem: φ is an isomorphism.

Proof: By the same construction, $\exists\, \psi : \mathcal{A}' \to \mathcal{A}$ homomorphism s.t.

$$\begin{tikzcd}
\mathcal{B}' \arrow[r, "f^{-1}"] \arrow[d, hook, "j'"'] \arrow[dr, "j\circ f^{-1}"] & \mathcal{B} \arrow[d, hook, "j"] \\
\mathcal{A}' \arrow[r, "\psi"'] & \mathcal{A}
\end{tikzcd}$$

Lemma: $\psi \circ \varphi : \mathcal{A} \to \mathcal{A}$ is the identity.

Proof: We have the commutative diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "f"] \arrow[d, hook, "j"'] \arrow[rr, bend left, "id"] & \mathcal{B}' \arrow[r, "f^{-1}"] \arrow[d, hook, "j'"'] & \mathcal{B} \arrow[d, hook, "j"] \\
\mathcal{A} \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\varphi"'] & \mathcal{A}' \arrow[r, "\psi"] & \mathcal{A}
\end{tikzcd}$$

$\psi \circ \varphi : \mathcal{A} \to \mathcal{A}$ is a homomorphism s.t. we have a commutative diagram

Commutative diagram (as in definition)

$$\begin{tikzcd} B \arrow[rr, hook, "j"] \arrow[dr, hook, "j"'] & & A \\ & A \arrow[ur, "\psi \circ \varphi"'] & \end{tikzcd}$$

, $\psi \circ \varphi : A \to A$ is a homomorphism

- But, $id : A \to A$ also gives the same diagram.

Hence, by uniqueness of the homomorphism in the definition, $\psi \circ \varphi = id$. □

- Similarly $\varphi \circ \psi = id$, thus φ & ψ are isomorphisms

Theorem: An abelian group A is free with basis $B \subseteq A$ iff every $x \in A$ is a unique linear combination of elements in B.

(Excercise: Details)

Proof: · Suppose A is free with basis B. Then $A \cong \mathbb{Z}[B]$, for which each element is a unique linear combination. We deduce for A.

· Conversely, if every $x \in A$ is uniquely $\sum_{i=1}^{n} \alpha_i b_i, b_i \in B$ we get an isomorphism $A \cong \mathbb{Z}[B], x = \sum_{i=1}^{n} \alpha_i b_i \mapsto \sum_{i=1}^{n} \alpha_i b_i$. As $\mathbb{Z}[B]$ is free, so is A.

R-modules: ('Vector spaces' with scalars in a ring)

- R is a ring (with identity)
- An (left) R-module M is an abelian group with

a scalar multiplication $R \times M \to M$, $(r, m) = r \cdot m$

s.t $\begin{cases} 1 \cdot m = m, \\ (r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m) \\ r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2 \\ (r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m \end{cases}$ (as in vector spaces)

Examples: $R = k$ field, R-module $\Leftrightarrow$ k-vector space.

- $R = \mathbb{Z}$, R-modules $\Leftrightarrow$ Abelian groups A,

with $n \cdot a = \underbrace{a + a \dots + a}_{n \text{ times}}$

Free R-modules: (· Homomorphisms like linear transformations }

- Same definition as free abelian groups.
- Given $\mathcal{B}$, we get a free R-module

$$R[\mathcal{B}] = \{ f : \mathcal{B} \to R \text{ fn: } f \text{ finitely supported} \}$$

- We have uniqueness up to isomorphism & characterisation in terms of unique linear combinations.

Simplicial chain complex

Let $\Delta = (V, S)$ be a simplicial complex

- We fix an 'orientation' on Δ, i.e, order vertices of each simplex in a consistent way.
- Hence a k-simplex in Δ is of the form

$$\langle v_0, \dots, v_k \rangle, \quad v_i \in V. \ (\{v_0, \dots, v_k\} \in S, v_0 < v_1 < \dots < v_k)$$

- $C_k(\Delta) =$ free abelian group with basis the k-simplices of Δ.

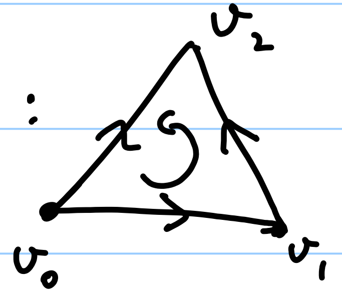
$C_k(\Delta ; \mathbb{Z}/_{2\mathbb{Z}}) = \mathbb{Z}/_2$-vector space with basis the k-simplices

Boundary: $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ homomorphism

We define ∂_k on a basis element $\langle v_0, v_1, \dots, v_k \rangle$ by

$$\partial_k : \langle v_0, \dots, v_k \rangle \mapsto \sum_{i=0}^{k} (-1)^i \langle v_0, \dots, \underset{\text{delete}}{\hat{v}_i}, \dots, v_k \rangle \in C_{k-1}(\Delta)$$

E.g. ∂ :



$$\longmapsto \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle$$

$$\langle v_0, v_1 \rangle = \langle v_0, v_1, \hat{v}_2 \rangle \qquad \langle v_0, v_2 \rangle = \langle v_0, \hat{v}_1, v_2 \rangle$$

Propn: $\partial_{k-1} \circ \partial_k = 0$

31/1/11

Lecture 7:

Propn: $\partial_{k-1} \circ \partial_k \equiv 0$

Pf: $\partial_{k-1} \circ \partial_k (\langle v_0, \ldots, v_k \rangle) = \partial_{k-1} \left(\sum_{i=0}^{k} (-1)^i \langle v_0, \ldots, \hat{v}_i, \ldots, v_k \rangle \right)$

$$= \sum_{i=0}^{k} (-1)^i \partial_{k-1} (\langle v_0, \ldots \hat{v}_i, \ldots, v_k \rangle)$$

$$= \sum_{i=0}^{k} (-1)^i \left[\sum_{j=0}^{i-1} (-1)^j \langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i \ldots \rangle + \sum_{j=i+1}^{k} (-1)^{j-1} \langle v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j \ldots \rangle \right]$$

$$= \sum_{\substack{i,j=0 \\ j<i}} \sum^{k} (-1)^{i+j} \langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots \rangle + \sum_{\substack{i,j=0 \\ j>i}} \sum^{k} (-1)^{i+j-1} \langle v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j \ldots \rangle$$

$$\parallel$$

$$\sum_{\substack{i,j=0 \\ j<i}} \sum^{k} (-1)^{i+j-1} \langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i \ldots \rangle$$

$$= 0$$

Hence, $\operatorname{im}(\partial_{k+1}) \subset \ker(\partial_k)$

Defn: $H_k(\Delta) = \ker(\partial_k) / \operatorname{im}(\partial_{k+1})$

Chain complexes:

- A chain complex (C_*, ∂_*) is a collection of free abelian groups C_k, $k \geq 0$, (or free R-modules), and a collection of homomorphisms $\partial_k : C_k \longrightarrow C_{k-1}$ s.t.
  $\partial_{k-1} \circ \partial_k = 0 \ \forall k \geq 1$
- Its homology is $H_k = \ker(\partial_k) / \operatorname{im}(\partial_{k+1})$

Singular homology:

[Topologists convention:
map = continuous function]

$\pi_1 \rightsquigarrow$ based on paths into a space X,

$$\alpha : \overset{=\Delta^1}{[0,1]} \longrightarrow X$$

- A singular n-simplex in a space X is a (continuous) map $\sigma : \Delta^n \longrightarrow X$, $\Delta^n = \langle e_0, \dots, e_n \rangle$ = std. n-simplex.
- $C_n(X) :=$ free abelian group with basis singular n-simplices in X.
- Elements of $C_n(X)$, which are called (singular) n-chains, are linear combinations of (singular) n-simplices.

Boundary homomorphism: $\partial_\cdot : C_n(X) \longrightarrow C_{n-1}(X)$

- We define this on a basis element, i.e., a singular n-simplex

$$\sigma : \Delta^n \longrightarrow X \quad , \quad \Delta^n = \langle e_0, \dots, e_n \rangle$$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{\underset{\langle e_0,\dots,e_{n-1}\rangle}{\underset{\wr}{\langle e_0,\dots,\hat{e}_i,\dots,e_n\rangle}}} \longrightarrow X$$

(by the canonical map)

Excercise: $\partial_{n-1} \circ \partial_n = 0 \quad \forall n \geq 1.$

- The singular homology of X, $H_*(X)$, is the homology of $(C_*(X), \partial_*)$.

· (Singular) homology $X \longrightarrow H_*(X)$ can be (axiomatically) characterised by:

We have a '<u>functor</u>' $X \longrightarrow H_*(X)$ satisfying 5 axioms:

(1) Exactness

(2) Excision

(3) Homotopy

(4) Dimension

(5) Compact support.

Functoriality: $\Big[\, \overset{\text{Spaces}}{\overset{\updownarrow}{\text{Objects}}} \to \overset{\text{Abelian gps}}{\overset{\updownarrow}{\text{Objects}}} \;;\; \overset{\text{Maps}}{\overset{\updownarrow}{\text{Morphisms}}} \to \overset{\text{homomorphisms}}{\overset{\updownarrow}{\text{Morphisms}}} \Big)$

- We can associate to a space X its homology groups $H_*(X) = \{H_n(X)\}_{n \geq 0}$
- We can associate to a map $f : X \to Y$ a collection of homomorphisms $f_* : H_n(X) \to H_n(Y)$.
- $id : X \to X$ induces $id_* = id : H_n(X) \to H_n(X)$
- $(f \circ g)_* = f_* \circ g_*$

Constructing f_* etc.

Given spaces X & Y and a map $f: X \to Y$, we construct homomorphisms $f_*: H_n(X) \to H_n(Y)$.

Step 1: Define homomorphisms on $C_*(X)$

$$f_\# : C_n(X) \longrightarrow C_n(Y) \text{ is defined by:}$$

given $\sigma: \Delta^n \longrightarrow X$,

$$f_\#(\sigma): \Delta^n \to Y \quad \text{is} \quad f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y.$$

Qn: Why do we have well-defined maps on homology?

Step 2: $f_\#$ is a chain homomorphism, i.e.,

$\partial_n^Y \circ f_\# = f_\# \circ \partial_n \quad \forall n$, or

$$\begin{tikzcd} C_n(X) \arrow[r, "f_\#"] \arrow[d, "\partial_n^X"] & C_n(Y) \arrow[d, "\partial_n^Y"] \\ C_{n-1}(X) \arrow[r, "f_\#"] & C_{n-1}(Y) \end{tikzcd}$$

commutes.

Pf : Exercise.

Step 3: Any chain homomorphism between chain complexes induces homomorphisms on homology groups.

<u>Thm:</u> Given $\varphi : (C_*, \partial_*) \to (C_*', \partial_*')$ a chain homomorphism, we have induced homomorphisms $\varphi_* : H_n \longrightarrow H_n'$ between the corresponding homology groups.

<u>Pf:</u> $H_n = \ker(\partial_n) / \mathrm{im}(\partial_{n+1})$, $H_n' = \ker(\partial_n') / \mathrm{im}(\partial_{n+1}')$

· First, we see that if $z_n \in \ker(\partial_n)$, then

$\varphi(z_n) \in \ker(\partial_n')$, for

$\partial_n' \circ \varphi(z_n) = \varphi(\partial_n z_n) = \varphi(0) = 0$

Hence we have a homomorphism

$\overline{\varphi} : \ker(\partial_n) \longrightarrow \ker(\partial_n') \longrightarrow \ker(\partial_n') / \mathrm{im}(\partial_{n+1}')$

- To see that this induces a map

$$\ker(\partial_n)/\mathrm{im}(\partial_{n+1}) \longrightarrow \ker(\partial_n')/\mathrm{im}(\partial_{n+1}')$$

it suffices to check that

$$b_n \in \mathrm{im}(\partial_{n+1}) \Rightarrow \bar{\varphi} : b_n \mapsto 0$$

- But $b_n \in \mathrm{im}(\partial_{n+1}) \Rightarrow b_n = \partial b_{n+1}$

$$\Rightarrow \varphi(b_n) = \varphi(\partial b_{n+1}) = \partial \varphi(b_{n+1}) \in \mathrm{im}(\partial_{n+1}')$$

$$\Rightarrow \bar{\varphi}(b_n) = 0 \in \ker(\partial_n')/\mathrm{im}(\partial_{n+1}').$$

2/2/2011

Lecture 8:

Singular homology is a functor:

Space $X \rightsquigarrow \{H_n(X)\}_{n\geq 0}$, H_n abelian gps

map: $f: X \to Y$ induces $f_*: H_n(X) \to H_n(Y)$ (homom.)

· $(f\circ g)_* = f_* \circ g_*$ and $(1_X)_* = 1$ (identity)

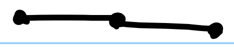
Propn: Suppose X & Y are homeomorphic then $H_n(X) \cong H_n(Y)$ $\forall n$.

Pf: If $f: X \to Y$ is a homeomorphism with inverse $g: Y \to X$, then $f_*: H_n(X) \to H_n(Y)$, $g_*: H_n(Y) \to H_n(X)$ and $f_* \circ g_* = (f\circ g)_* = (1)_* = \underline{1}$; $g_* \circ f_* = 1$. So f_* isomorphism.

Singular homology:

- Space $X \rightsquigarrow$ Chain complex $C_*(X) \to$ Homology $H_*(X)$
- Map $f: X \to Y \rightsquigarrow$ Chain homomorphism $f_\# \to$ Homomorphisms f_*.

Why not stop at chain complexes:

- Simplicial homology: the chain complex depends on the specific simplicial complex Σ, not on $|\Sigma|$. E.g.  vs 
- Singular homology: $C_*(X)$ is homeomorphism invariant

but $\begin{cases} \cdot \text{ Huge} \to \text{Uncountable basis} \\ \cdot \text{ Not homotopy invariant.} \end{cases}$

- $C_n(X)$ = singular n-chains ∈ Free abelian group with basis singular n-simplices $\sigma : \Delta^n \to X$
- Chain homomorphism: $f_\# : C_n(X) \to C_n(Y)$ homomorphism s.t.

$$\begin{tikzcd}
{} \arrow[d, dashed, "\partial"'] & {} \arrow[d, dashed, "\partial"] \\
C_2(X) \arrow[r, "f_\#"] \arrow[d, "\partial"'] & C_2(Y) \arrow[d, "\partial"] \\
C_1(X) \arrow[r, "f_\#"] \arrow[d, "\partial"'] & C_1(Y) \arrow[d, "\partial"] \\
C_0(X) \arrow[r, "f_\#"] & C_0(Y)
\end{tikzcd}$$

commutes

$f : X \to Y$ induces

$f_\# : \sigma \mapsto f \circ \sigma.$

$1_\# = 1$, $1_* = 1$

$(f \circ g)_\# = f_\# \circ g_\#$,

$(f \circ g)_* = f_* \circ g_*$.

- Such a chain homomorphism induces homomorphisms

$f_* : H_n(X) \longrightarrow H_n(Y)$

Dimension Axiom: Let $X = \{p\}$. Then

$$H_n(X) = \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n>0 \end{cases}$$

Pf: $C_n(X)$: Singular n-simplices $\sigma : \Delta^n \to X$.

Thus, $\exists!$ singular n-simplex σ_n.

- $C_n(X) = \mathbb{Z}[\sigma_n] \cong \mathbb{Z}$.
- $\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle} = \sum_{i=0}^{n} (-1)^i \cdot \sigma_{n-1}$

$$= \begin{cases} \sigma_{n-1} \text{ if } n \text{ even} \\ 0 \text{ if } n \text{ odd} \end{cases}$$

Thus, the chain complex of a point is

$$\begin{tikzcd}
C_3(X) \arrow[r, phantom, "="] & \mathbb{Z} \arrow[d, "\partial_3"', "0"] \\
C_2(X) \arrow[r, phantom, "="] & \mathbb{Z} \arrow[d, "\partial_2"', "1"] \\
C_1(X) \arrow[r, phantom, "="] & \mathbb{Z} \arrow[d, "\partial_1"', "0"] \\
C_0(X) \arrow[r, phantom, "="] & \mathbb{Z}
\end{tikzcd}$$

Homology:

$H_0(X) = \mathbb{Z}/_{im(0)} = \mathbb{Z}$

- $H_{2k+1}(X) = \ker(\partial_{2k+1})/_{im(\partial_{2k})}$
  $= \ker(0)/_{im(1)}$
  $= \mathbb{Z}/_{\mathbb{Z}} \cong 0$

- $H_{2k}(X) = \ker(\partial_{2k})/_{im(\partial_{2k+1})}$ $(k \geq 1)$
  $= \ker(1)/_{im(0)} = 0$

Homotopy Axiom: Suppose $f, g : X \to Y$ are maps such that f is homotopic to g $(f \sim g)$.

Then $f_* = g_* : H_n(X) \to H_n(Y)$.

Cor: If $X \overset{\text{h.e.}}{\simeq} Y$, then $H_*(X) \cong H_*(Y)$

Pf: If $f : X \to Y$ is a h.e., with $g : Y \to X$ s.t.

$$f \circ g \sim 1 \quad \& \quad g \circ f \sim 1.$$

Then $f_* \circ g_* = (f \circ g)_* \underset{\uparrow}{=} 1_* = 1$

as $f \circ g \sim 1$

$$\& \quad g_* \circ f_* = 1$$

$\square$

- X contractible $\implies H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0 & \text{otherwise} \end{cases}$

Exercise *: Does this together with $H_1(S^1) = \mathbb{Z}$ imply $H_i = \pi_1 / [\pi_1, \pi_1]$?

- Suppose $f, g : X \to Y$, $f \sim g$.

Then we see $f_\#$ & $g_\#$ are 'chain homotopic'.

$\{\varphi_n\}$ $\{\psi_n\}$

Defn: $\varphi, \psi : (C_*, \partial_*) \longrightarrow (C'_*, \partial'_*)$, chain homomorphisms, are chain homotopic if $\exists\, H_n : C_n \longrightarrow C'_{n+1}$ homomorphism, $\forall n \geq 0$

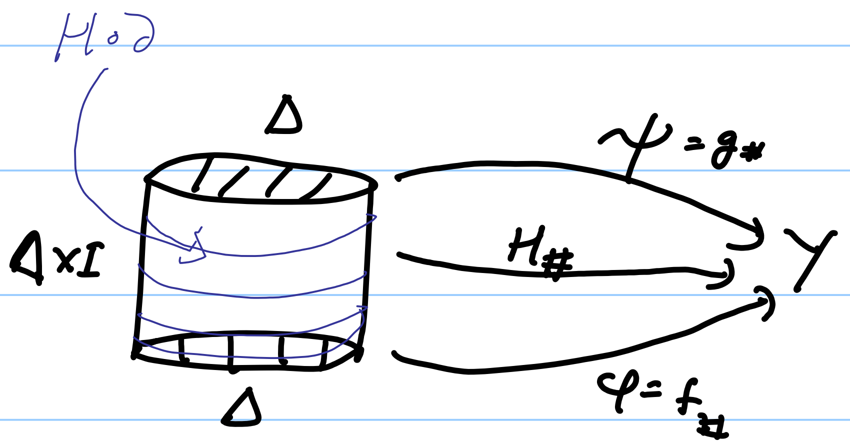
s.t. $\partial'_{n+1} H_n + H_{n-1} \partial_n = \varphi_n - \psi_n$.

i.e. $\partial' H + H \partial = \varphi - \psi$

We have:

$$\begin{tikzcd}
C_3 \arrow[r, "\varphi_{,}\psi"] \arrow[d, "\partial_3"'] & C_3' \arrow[d, "\partial_3'"] \\
C_2 \arrow[r, "\varphi_{,}\psi"] \arrow[d, "\partial_2"'] \arrow[ur, "H_2"] & C_2' \arrow[d, "\partial_2'"] \\
C_1 \arrow[r, "\varphi_1{,}\psi_1"] \arrow[d, "\partial_1"'] \arrow[ur, "H_1"] & C_1' \arrow[d, "\partial_1'"] \\
C_0 \arrow[r, "\varphi_0{,}\psi_0"'] \arrow[ur, "H_0"] & C_0'
\end{tikzcd}$$

$$\boxed{\partial' H + H\partial = \varphi - \psi}$$



- H raises degree because we product with I.
- $\partial(\Delta \times I) = (\Delta \times \{0\}) \cup (\Delta \times \{1\}) \cup (\partial\Delta \times I)$

$\partial' H$ on $\partial(\Delta \times I) \rightsquigarrow$ φ ψ $H \circ \partial$

Lemma: If $\varphi, \psi : (C_*, \partial_*) \to (C_*', \partial_*')$ are chain homotopic, then $\varphi_*, \psi_* : H_n \longrightarrow H_n'$ are equal $\forall\, n \geq 0$.

Pf: Let $z_n \in \ker(\partial_n) = Z_n$ be an 'n-cycle'

. Then $[z_n] \in H_n' = \ker(\partial_n) / \mathrm{im}(\partial_{n+1}) = Z_n / B_n \to$ boundaries

To show: $\varphi(z_n)$ & $\psi(z_n)$ represent the same element of $H_n' = Z_n' / B_n' = Z_n' / \mathrm{im}(\partial_{n+1}')$

i.e., $\varphi(z_n) - \psi(z_n) \in \mathrm{im}(\partial_{n+1}')$. ('are homologous')

$$\varphi(z_n) - \psi(z_n) = \partial_{n+1}' H_n(z_n) + \underbrace{H_{n-1}\partial_n(z_n)}_{0}, \quad \left\{ \begin{array}{l} H \text{ chain homotopy} \\ \text{from } \varphi \text{ to } \psi. \end{array} \right.$$

$$\in \mathrm{im}(\partial_{n+1}')$$

What remains: Given $f, g : X \longrightarrow Y$, we want

$$H_n : C_n(X) \longrightarrow C_{n+1}(Y) \text{ homomorphisms, } n \geq 0.$$

$$\partial^Y_{n+1} H_n + H_{n-1} \partial^X_n = f_{\# n} - g_{\# n}.$$

- Suffices to define H_n on basis elements, i.e.

$$\sigma : \Delta^n \longrightarrow X$$

- $f_\#(\sigma) = f \circ \sigma$, $g_\#(\sigma) = g \circ \sigma$.
- We have $\Delta^n \times I \xrightarrow{\sigma \times \mathbb{1}} X \times I \xrightarrow{H} Y$

- We would like $H_\#(\sigma) = H(\sigma \times \mathbb{1})$.

7/2/2011

Lecture 9

An observation:

- A singular n-simplex is a map

  $\sigma : \Delta^n \longrightarrow X$, $\quad \sigma \in C_n(X)$

- $\sigma : \Delta^n \to X$ is a map, so $\sigma_\# : C_n(\Delta^n) \to C_n(X)$
- $1 : \Delta^n \longrightarrow \Delta^n$ gives an element $1 \in C_n(\Delta^n)$
- $\sigma_\#(\underset{\substack{\uparrow \\ C_n(\Delta^n)}}{1}) = \sigma \circ 1 = \underset{\substack{\uparrow \\ C^n(X)}}{\sigma}$,

- We often construct elements of $C_n(\Delta)$ etc. and consider $\sigma_\#(\cdot)$.

Lemma: $f, g : X \to Y$ are homotopic maps, then $f_\#, g_\# : C_*(X) \to C_*(Y)$ are chain homotopic chain homomorphisms.

Pf: Let $H : X \times [0,1] \to Y$ be a homotopy from f to g.

We construct $\hat{H} : C_*(X) \to C_{*+1}(Y)$ s.t.

$$\partial^Y_* \hat{H} + \hat{H} \partial^X_{*+1} = f_\# - g_\#$$

- $\hat{H} : C_n(X) \to C_{n+1}(Y)$ is defined on

$$\sigma : \Delta^n \to X$$

using $H : X \times I \to Y$

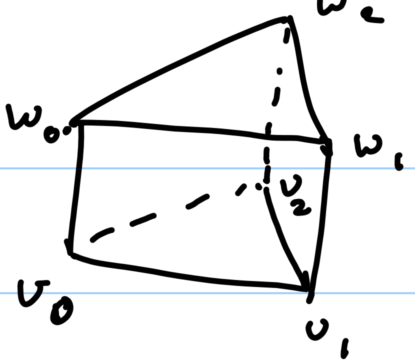
- We have

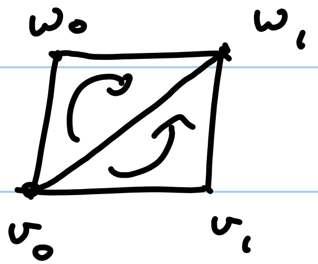
$$\Delta^n \times I \xrightarrow{\sigma \times 1\!\!1} X \times I \xrightarrow{H} Y$$

so $\quad C_{n+1}(\Delta^n \times I) \xrightarrow{(\sigma \times 1\!\!1)_\#} C_{n+1}(X \times I) \xrightarrow{H_\#} C_{n+1}(Y)$

- We would like to apply this to '$1 \in C_{n+1}(\Delta^n \times I)$', but this is not a singular n-simplex.
- Instead, we write $\Delta^n \times I$ as a 'n-chain'.
- Recall $\Delta^n = \langle e_0, \dots, e_n \rangle$.
- Let $v_i = (e_i, 0)$, $w_i = (e_i, 1)$,

then $\Delta^n \times I$ is the convex hull of v_i & w_i





Exercise: $\Delta^n \times I$ is the union of simplices $\tau_i = \langle v_0, \dots, v_i, w_i, \dots, w_n \rangle$, $i = 0, \dots, n$, and these have disjoint interiors.

- We define $\hat{H} = P : C_n(X) \longrightarrow C_{n+1}(Y)$ by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times \mathbb{I}) |_{\underset{\Delta^{n+1}}{\overset{\cong}{\langle v_0, \dots, v_i, w_i, \dots, w_n \rangle}}}$$

Exercise: $\partial P + P \partial = f_\# - g_\# \quad [f = H|_{X \times \{0\}}, g = H|_{\dots}]$

Thus, $\hat{H} = P$ gives a chain homotopy from $f_\#$ to $g_\#$.

- $f \sim g \Rightarrow f_\# \underset{\text{chain homotopic}}{\sim} g_\# \Rightarrow f_* = g_*$

This is the <u>homotopy axiom</u>

- Two state the remaining axioms, we need <u>relative homology.</u>

Defn: Let (X, A) be a pair of topological spaces, i.e., $\cdot\, X$ is a topological space

$\cdot\, A$ is a subspace.

- Then $C_*(A) \subset C_*(X)$
- We define $C_n(X,A) = C_n(X)/C_n(A)$, $n \geq 0$.

Propn: $(C_*(X,A), \partial_*)$ is a chain complex.

Pf: $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ satisfies $\partial_n(C_n(A)) \subset C_{n-1}(A)$, hence we have induced homomorphisms

$$\partial_n : \underset{\underset{C_n(X,A)}{\shortparallel}}{\frac{C_n(X)}{C_n(A)}} \longrightarrow \frac{C_{n-1}(X)}{C_{n-1}(A)} \, , \quad = C_{n-1}(X,A)$$

- $\partial_{n-1} \circ \partial_n = 0$ follows from the statement for $C_n(X)$.
- Finally $C_n(X,A)$ is a free abelian group as a basis $\mathcal{B}_A$ for $C_n(A)$ is a subset of a basis $\mathcal{B}_X$ for $C_n(X)$, so $C_n(X,A)$ is free with basis $\mathcal{B}_X \setminus \mathcal{B}_A$. (Exercise *)

$\square$

__Defn:__ The relative homology $H_*(X,A)$ is the homology of $C_*(X,A)$

What is homology?

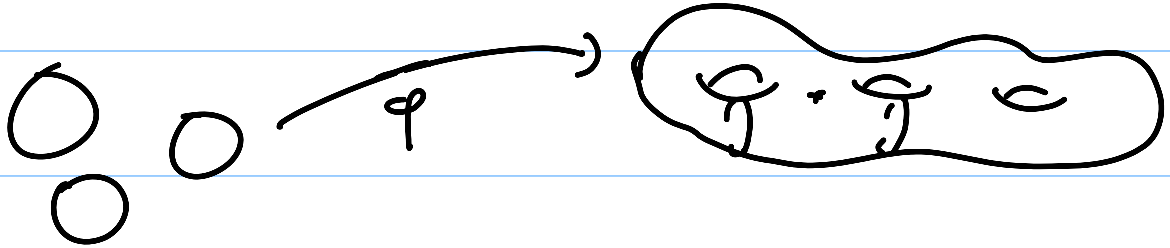
$\pi_1 = \{\text{based maps } S^1 \longrightarrow X\}/\sim$ = $H: \underset{\text{based}}{S^1 \times I} \to X$

- To drop basepoints, we abelianise

(**Theorem:** (Not proved, see Hatcher)

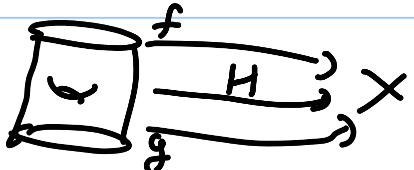
$$H_1 = \pi_1 / [\pi_1, \pi_1] = \text{abelianisation of } \pi_1$$)

- $H_1 : \{\text{maps from } \coprod S^1 \longrightarrow X.\}/\sim$



Here $\sim$ is in terms of $H : (\Sigma, \partial\Sigma) \longrightarrow X$

Homotopy:

How do we generalise maps from S^1?

$\pi_2(X,x)$

$\|$

maps from S^2
(based)

$H_2(X) = \{f : \Sigma \longrightarrow X\}/\sim$

$\|$

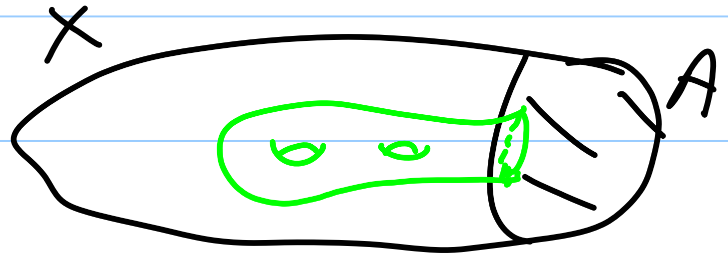
maps from Surfaces: Compact,
without ∂

- This is 'nicer'

- $H_2(X,A) = \{f : (\Sigma, \partial\Sigma) \longrightarrow (X,A)\}/\sim$

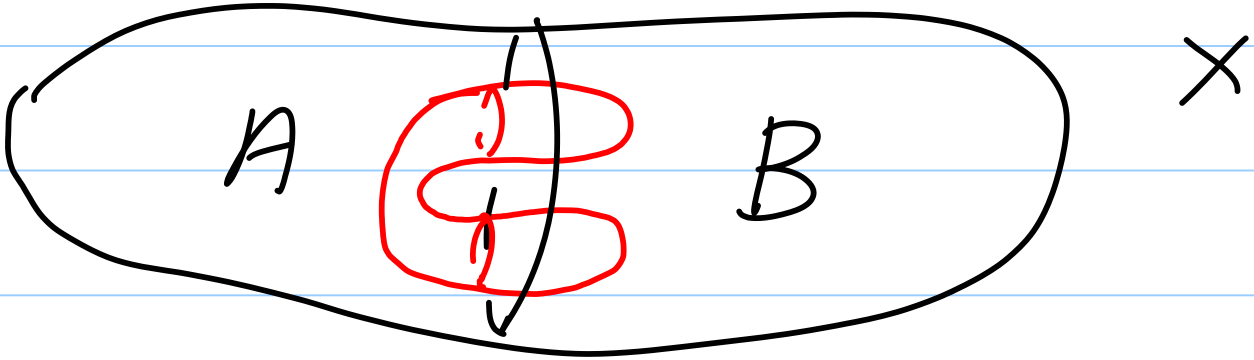
$\|$

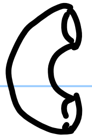
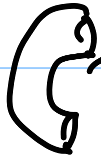
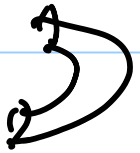
maps from compact surface with ∂



Why is homology easier?

- We have 'combination theorems'



- If we take $f: S^2 \longrightarrow X$ and restrict to A & B, we get $f:$  $\longrightarrow A$ (planar surface)
- When we glue maps $f:$  $\longrightarrow A$ & $g:$  $\longrightarrow B$

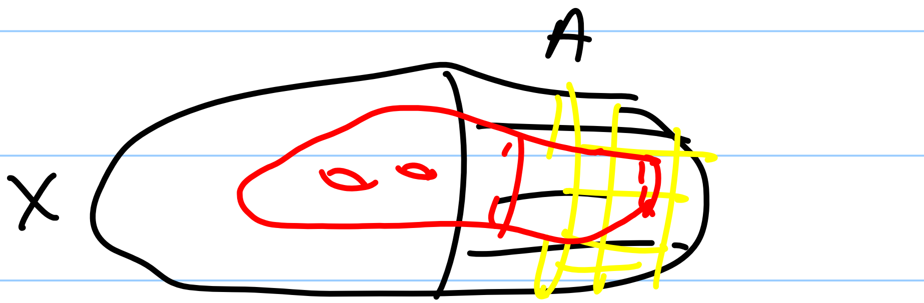
we get  $\longrightarrow X$.

Excision Axiom: Given (X, A), suppose $B \subset A$ is a subspace such that $\bar{B} \subset \mathring{A}$, then

$$i : (X \setminus B, A \setminus B) \longrightarrow (X, A) \quad \text{inclusion}$$

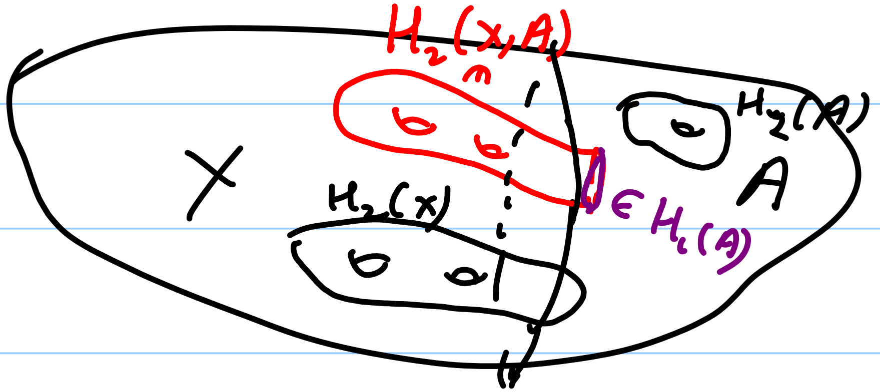
induces an isomorphism

$$i_* : H_n(X \setminus B, A \setminus B) \to H_n(X, A) \quad \forall n \geq 0$$



Exact sequence:

$$\dots \to H_{n+1}(X,A) \to H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \dots$$



9/2/2011

Lecture 10: Exact Sequences

A sequence of homomorphisms between Abelian groups

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \to \dots \xrightarrow{\varphi} A_n \quad \text{(could be infinite)}$$

is exact if $\ker(\varphi_{i+1}) = \mathrm{im}(\varphi_i)$ $\forall i$, $1 \leq i \leq n-1$.

E.g. A chain complex

$$\cdots \to C_2 \to C_1 \to C_0$$

is an exact sequence iff $H_i = 0$ $\forall i \geq 1$

E.g.
- $0 \to A \xrightarrow{\varphi} B$ is exact iff φ is injective
- $A \xrightarrow{\varphi} B \to 0$ is exact iff φ is surjective
- $0 \longrightarrow A \xrightarrow{\varphi} B \to 0$ exact iff φ isomorphism.

Short exact Sequences: (s.e.s)

- An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is called a short exact sequence.

- $A \hookrightarrow B$, $B \twoheadrightarrow C$

- In particular, $C = B/_{\alpha(A)}$, $\alpha(A) \cong A$

E.g. (a) $0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow \mathbb{Z}/_{p\mathbb{Z}} \longrightarrow 0$

$$n \longmapsto (n, 0)$$

(b) $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}/_{p\mathbb{Z}} \longrightarrow \mathbb{Z}/_{p\mathbb{Z}} \longrightarrow 0$

$$(n, m) \longmapsto m$$

General Qn:

Given A & C, classify s.e.s. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

__Five Lemma:__ Given a commutative diagram of Abelian groups with exact rows (i.e. $A\to B\to C\to D\to E$ & ... exact sequences)

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"', "\simeq"] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"', "\simeq"] & E \arrow[d, hook, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

with α onto, β, δ isomorphisms and ε injective, γ is an isomorphism.

__Pf:__ A diagram chase.

First we show γ is injective.

$$\begin{tikzcd}
a & b \arrow[r, mapsto] & \in c \arrow[r, mapsto] & & d & \\
A \arrow[r] \arrow[d, two heads, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"', "\cong"] & C \arrow[rr] \arrow[d, "\gamma"'] & c \arrow[d, mapsto] & D \arrow[r] \arrow[d, "\delta"', "\cong"] & E \arrow[d, hook, "\epsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[rr] & 0 \mapsto 0 & D' \arrow[r] & E' \\
a' \arrow[r, mapsto] & b' \arrow[r, mapsto] & 0 & & &
\end{tikzcd}$$

- Suppose $c \in C$, $\gamma(c) = 0$. Let $c \mapsto d \in D$
- Now $d \mapsto 0 \in D'$ (by $\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ C' \arrow[r] & D' \end{tikzcd}$ commuting) $\Rightarrow d = 0$
- Thus $c \mapsto 0 \in D \Rightarrow \exists b \in B$, $b \mapsto c$
- Let $\beta : b \mapsto b'$; $b' \mapsto 0 \in C' \Rightarrow \exists a' \in A'$, $a' \mapsto b'$
- α onto $\Rightarrow \exists a \in A$, $\alpha : a \mapsto a'$

Claim: $a \mapsto b \in B$.

· Let $a \mapsto b_1 \in B$; <u>Claim</u>: $b_1 = b$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"', "\simeq"] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"', "\simeq"] & E \arrow[d, hook, "\epsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

$$\begin{tikzcd}
 & & b_1 & \\
a \arrow[urr, mapsto] \arrow[d, mapsto] & & b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[d, mapsto] \\
a' \arrow[rr, mapsto] & & b' \arrow[r, mapsto] & 0
\end{tikzcd}$$

· $b_1 \mapsto b_1' \in B'$, $b_1' = b'$ by

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[ddd, mapsto] & b_1 & \\
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, mapsto] & \\
A' \arrow[r] & B' & \\
a' \arrow[r, mapsto] & b' & b_1'
\end{tikzcd}$$

· As β is 1-1, $b_1 = b$.

Thus, $a \mapsto b \mapsto c$. By exactness, $c = 0$ as required.

This shows that γ is one-to-one.

$\underline{\gamma \text{ is onto}}$: Let $c' \in C'$

$$\begin{tikzcd}
 & b & c_1 \arrow[r, mapsto] & d \arrow[r, mapsto] & e \\
A \arrow[r] \arrow[d, two heads, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"', "\cong"] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"', "\cong"] & E \arrow[d, hook, "\epsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \\
 & b' & c_1', \hat{c}' \arrow[r, mapsto] & d' \arrow[r, mapsto] & 0
\end{tikzcd}$$

- Let $c' \mapsto d'$; Note $d' \mapsto 0$.
- As δ is an isomorphism, $\exists\, d \in D$, $\delta : d \mapsto d'$
- Let $d \mapsto e \in E$; $e \mapsto 0 \in E' \Rightarrow e = 0$
- Thus $d \mapsto 0 \Rightarrow \exists\, c_1 \in C$, $c_1 \mapsto d$
- Let $\gamma : c_1 \mapsto c_1'$; $c_1' \mapsto d' \Rightarrow c_1' - c' \mapsto 0$
- Hence $\exists\, b' \in B$, $b' \mapsto c_1' - c'$; $\exists\, b \in B, \beta : b \mapsto b'$.

Strategy: Find $c_1 \in C$ & modify

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"', "\cong"] & \overset{c_1}{C} \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"', "\cong"] & E \arrow[d, hook, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & \underset{c_1', c'}{C'} \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, mapsto] \arrow[d, mapsto] & c_2 \arrow[d, mapsto] & \\
b' \arrow[r, mapsto] & c_1' - c' \arrow[r, mapsto] & 0
\end{tikzcd}$$

- Let $b \mapsto c_2 \in C$, then $\gamma : c_2 \mapsto c_1' - c'$
- Hence $\gamma(c_1 - c_2) = \gamma(c_1) - \gamma(c_2)$

$$= c_1' - (c_1' - c') = c'$$

- Thus, if $c = c_1 - c_2$, $c \mapsto c'$ as required.

$\square$

Ref: Manin: Mathematics : . _ _ _

Zig-Zag lemma: A short exact sequence of chain complexes

$$0 \longrightarrow C''_* \xrightarrow{i_\#} C_* \xrightarrow{j_\#} C'_* \longrightarrow 0$$

induces a long exact sequence of homology groups

$$\cdots \longrightarrow H'_{n+1} \longrightarrow H''_n \xrightarrow{i_*} H_n \xrightarrow{j_*} H'_n \xrightarrow{\delta} H''_{n-1} \xrightarrow{i_*} H_{n-1} \xrightarrow{j_*}$$

S.e.s. of chain complexes: $i_\#$ & $j_\#$ are chain homomorphism s.t. for each n,

$$0 \longrightarrow C''_n \xrightarrow{i_\#} C_n \longrightarrow C'_n \longrightarrow 0$$

is an exact sequence.

Construction of $\delta : H_n' \longrightarrow H_{n-1}''$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}'' \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & C_{n+1}' \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & C_n'' \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_n' \arrow[r] \arrow[d] & 0 & z_n \quad z_n' \arrow[d, mapsto] \\
0 \arrow[r] & C_{n-1}'' \arrow[r] & C_{n-1} \arrow[r] & C_{n-1}' \arrow[r] & 0 & 0 \\
 & z_{n-1}'' & z_{n-1} & & &
\end{tikzcd}$$

- Suppose $[z_n'] \in H_n'$, $\partial z_n' = 0$
- $\exists\ z_n \in C_n$, $z_n \longmapsto z_n'$; $z_{n-1} = \partial z_n \in C_{n-1}$
- $z_{n-1} \longmapsto 0 \Rightarrow \exists\ z_{n-1}''$ s.t. $z_{n-1}'' \longmapsto z_{n-1}$
- **Claim:** $\partial z_{n-1}'' = 0$

Claim: $\partial z''_{n-1} = 0$

Pf: • $\partial z_{n-1} = \partial^2 z_n = 0$

$$\begin{tikzcd}
0 \arrow[r] & C''_n \arrow[r] \arrow[d] & C_n \arrow[d] & \in z_n \arrow[d, mapsto] \\
0 \arrow[r] & C''_{n-1} \in z''_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[d] & z_{n-1} = \partial z_n \arrow[d, mapsto] \\
0 \arrow[r] & C''_{n-2} \arrow[r] & C_{n-2} & \partial^2 z_n = 0 \\
 & \partial z''_{n-1} \arrow[u] & &
\end{tikzcd}$$

• $\partial z''_{n-1} \mapsto 0 \implies \partial z''_{n-1} = 0$ by injectivity.

• Thus, an element $[z_n'] \in H_n'$ gives an element $[z''_{n-1}] \in H''_{n-1}$.

• However, this depends on choices: $\begin{cases} \cdot \text{ representative } z_n' \\ \cdot \text{ pullback } z_n \end{cases}$

Independence of choice of z_n:

- Suppose $\hat{z}_n \mapsto z_n'$, $\partial \hat{z}_n = \hat{z}_{n-1}$ & $\hat{z}''_{n-1} \mapsto \hat{z}_{n-1}$

$$\begin{tikzcd}
0 \arrow[r] & C''_{n+1} \arrow[r] \arrow[d, swap, "y''_n"] & C_{n+1} \arrow[r] \arrow[d, "z_n"] & C'_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C''_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C'_n \arrow[r, "z_n'"] \arrow[d] & 0 \\
0 \arrow[r] & C''_{n-1} \arrow[r] & C_{n-1} \arrow[r] & C'_{n-1} \arrow[r] & 0
\end{tikzcd}$$

To show: $[z''_{n-1}] = [\hat{z}''_{n-1}]$, i.e. $z''_{n-1} - \hat{z}''_{n-1} \in \operatorname{im}(\partial)$

- $z_n - \hat{z}_n \mapsto 0 \Rightarrow \exists\, y''_n \in C''_n$, $y''_n \mapsto z_n - \hat{z}_n$
- **Claim:** $\partial y''_n = z''_{n-1} - \hat{z}''_{n-1}$

$$\left.\begin{aligned} \partial y''_n &\mapsto \partial(z_n - \hat{z}_n) = z_{n-1} - \hat{z}_{n-1} \ \& \\ z''_{n-1} - \hat{z}''_{n-1} &\mapsto z_{n-1} - \hat{z}_{n-1} \end{aligned}\right\} \text{Use injectivity.}$$

14/2/2011

Lecture 11 :

Proof of Zig-Zag lemma (contd.)

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}'' \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & C_{n+1}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n'' \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_n' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}'' \arrow[r] & C_{n-1} \arrow[r] & C_{n-1}' \arrow[r] & 0
\end{tikzcd}$$

Step 1: Definition of $\delta : \underset{Z_n'/B_n'}{\underset{\parallel}{H_n'}} \longrightarrow \underset{Z_{n-1}''/B_{n-1}''}{\underset{\parallel}{H_{n-1}''}}$

(a) Define $\tilde{\delta} : Z_n' \longrightarrow C_{n-1}''$ (with choices)

(b) Show image$(\tilde{\delta}) \in Z_{n-1}''$, giving $\delta : Z_n' \to Z_{n-1}'' \to H_{n-1}''$

(c) Show $\delta : Z_n' \to H_{n-1}''$ is well-defined

(d) Show $\delta(B_n') = 0$ (i.e. $\tilde{\delta}(B_n') \subset B_{n-1}''$)

(a) Given $z_n' \in Z_n'$, define $\tilde{\delta}(z_n')$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}'' \arrow[r, "i"] \arrow[d, "''"] & C_{n+1} \arrow[r, "j"] \arrow[d, "z_n"] & C_{n+1}' \arrow[r] \arrow[d, "'\ z_n'"] & 0 \\
0 \arrow[r] & C_n'' \arrow[r, "i"] \arrow[d, "''"] & \overset{\wedge}{C_n} \arrow[r, "j"] \arrow[d, "\partial"] & C_n' \arrow[r] \arrow[d, "'"] & 0 \\
0 \arrow[r] & C_{n-1}'' \arrow[r, "i"] & \underset{\underset{z_{n-1}}{\uparrow}}{C_{n-1}} \arrow[r, "j"] & C_{n-1}' \arrow[r] & 0
\end{tikzcd}$$

- $j: z_n \longmapsto z_n'$; $z_{n-1} = \partial z_n$
- $j(z_{n-1}) = \partial z_n' = 0 \Rightarrow \exists\, z_{n-1}'' \in C_{n-1}''$, $i(z_{n-1}'') = z_{n-1}$
- z_n is <u>not</u> canonical.
- Fixing z_n, as i is injective z_{n-1}'' is determined.
- $\tilde{\delta}(z_n') := z_{n-1}''$

(b) z''_{n-1} is a cycle, i.e., $\partial z''_{n-1} = 0$

$\cdot i : \partial z''_{n-1} \longmapsto \partial z_{n-1} = \partial^2 z_n = 0$

$$\begin{tikzcd}
0 \arrow[r] & C''_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d, "z_n"'] & C'_{n+1} \arrow[r] \arrow[d, "z'_n"'] & 0 \\
0 \arrow[r] & C''_n \arrow[r] \arrow[d, "z''_{n-1}"'] & C_n \arrow[r] \arrow[d] & C'_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C''_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d, phantom, "z_{n-1}"] & C'_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C''_{n-2} \arrow[r] & C_{n-2} \arrow[r] & C'_{n-2} \arrow[r] & 0
\end{tikzcd}$$

$\partial z'_{n-1} \longmapsto \partial z_{n-1}$

- As i is 1-1, $\partial z''_{n-1} = 0$
- Thus, $\tilde{\delta} z'_n \in Z''_{n-1}$, so

$$\begin{tikzcd}
Z'_n \arrow[r, "\tilde{\delta}"] \arrow[rr, bend right, "\delta"'] & Z''_{n-1} \arrow[r] & H''_{n-1}
\end{tikzcd}$$

(c) $\delta : z_n \longrightarrow H''_{n-1}$ is well-defined.

$$\begin{tikzcd}
0 \arrow[r] & C''_{n+1} \arrow[r] \arrow[d, "''"] & C_{n+1} \arrow[r] \arrow[d] & C'_{n+1} \arrow[r] \arrow[d, "'"] & 0 \\
0 \arrow[r] & \overset{b''_n}{C''_n} \arrow[r] \arrow[d, "''"] & \overset{\hat{z}_n \quad z_n}{C_n} \arrow[r, "j"'] \arrow[d] & \overset{z'_n}{C'_n} \arrow[r] \arrow[d, "'"] & 0 \\
0 \arrow[r] & C''_{n-1} \arrow[r] & C_{n-1} \arrow[r] & C'_{n-1} \arrow[r] & 0 \\
 & \hat{z}''_{n-1}, z''_{n-1} & \hat{z}_{n-1} \quad z_{n-1} & &
\end{tikzcd}$$

- We need to show that if we took $\hat{z}_n$ in place of z_n, we get the same element in H''_n

- $j : \hat{z}_n - z_n \longmapsto z'_n - z'_n = 0 \Rightarrow \exists\, b''_n,\ b''_n \longmapsto \hat{z}_n - z_n$

- $i(\partial b''_n) = \hat{z}_{n-1} - z_{n-1}$ & $i(\hat{z}''_{n-1} - z''_{n-1}) = \hat{z}_{n-1} - z_{n-1}$

- As i is 1-1, $\partial b''_n = \hat{z}_{n-1} - z_{n-1}$

$\Rightarrow [\hat{z}_{n-1}] = [z_{n-1}] \in H''_n$, i.e., $\delta : z'_n \to H''_{n-1}$ well-defined

(d) $\delta(\partial b'_{n+1}) = 0$

$$\begin{tikzcd}
0 \arrow[r] & C''_{n+1} \arrow[r] \arrow[d, "''"] & \overset{b_{n+1}}{C_{n+1}} \arrow[r] \arrow[d] & \overset{b'_{n+1}}{C'_{n+1}} \arrow[r] \arrow[d, "'"] & 0 \\
0 \arrow[r] & C''_n \arrow[r] \arrow[d, "''"] & \overset{z_n}{C_n} \arrow[r] \arrow[d] & \overset{\partial b'_{n+1} = z'_n}{C'_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C''_{n-1} \arrow[r] & \underset{0 = z_{n-1}}{C_{n-1}} \arrow[r] & C'_{n-1} \arrow[r] & 0
\end{tikzcd}$$

- $\exists\, b_{n+1}$, $j : b_{n+1} \mapsto b'_{n+1}$
- Let $z_n = \partial b_{n+1}$, $j : z_n \mapsto z'_n$ as reqd.
- Then $z_{n-1} = \partial z_n = \partial^2 b_{n+1} = 0$

$\Rightarrow$ $z''_{n-1} = 0$.

Thus, $\delta(\partial b'_{n+1}) = 0$.

□

$$H'_{n+1} \xrightarrow{\delta} H''_n \xrightarrow{i_*} H_n \xrightarrow{j_*} H'_n \xrightarrow{\delta} H''_{n-1} \longrightarrow$$

Step 2: Exactness at H_n.

- $j_* \circ i_* = 0$ as $j_\# \circ i_\# = 0$

$$\begin{tikzcd}
0 \arrow[r] & C''_{n+1} \arrow[r] \arrow[d, "''"] & \overset{b_{n+1}}{C_{n+1}} \arrow[r, "\longmapsto"] \arrow[d] & \overset{b'_{n+1}}{C'_{n+1}} \arrow[r] \arrow[d, "'"] \arrow[d, mapsto, bend right] & 0 \\
0 \arrow[r] & \overset{z''_n}{C''_n} \arrow[r] \arrow[d, "''"] & \overset{\hat{z}_n}{C_n} \arrow[r, "z_n \longmapsto"] \arrow[d] & \overset{z'_n}{C'_n} \arrow[r] \arrow[d, "'"] & 0 \\
0 \arrow[r] & \overset{\partial z''_n}{C''_{n-1}} \arrow[r] & C_{n-1} \arrow[r] & C'_{n-1} \arrow[r] & 0
\end{tikzcd}$$

- Suppose $[\overset{\in Z_n}{z_n}] \in \ker(j_*) \Rightarrow j(z_n) =: z'_n = \partial b'_{n+1}$
- $\exists b_{n+1}$, $j : b_{n+1} \mapsto b'_{n+1}$; let $\hat{z}_n = z_n - \partial b_{n+1}$
- $[\hat{z}_n] = [z_n] \in H_n$ & $j(\hat{z}_n) = z'_n - z'_n = 0$.
- $\exists z''_n \in C''_n$, $i(z''_n) = \hat{z}_n$.
- $i(\partial z''_n) = \partial \hat{z}_n = 0$; Thus, $i_*(\overset{H''_n}{\overset{\downarrow}{[z''_n]}}) = [\hat{z}_n] = [z_n]$ $\square$

$$H_{n+1}' \xrightarrow{\delta} H_n'' \xrightarrow{i_*} H_n \xrightarrow{j_*} H_n' \xrightarrow{\delta} H_{n-1}''$$

Step 3: Exactness at H_n'.

(a) $\delta \circ j_*([z_n]) = 0 \quad \forall [z_n] \in H_n$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}'' \arrow[r] \arrow[d, "b_n''"'] & C_{n+1} \arrow[r] \arrow[d] & C_{n+1}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n'' \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_n' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}'' \arrow[r] & C_{n-1} \arrow[r] & C_{n-1}' \arrow[r] & 0
\end{tikzcd}$$

z_n (at C_n), z_n' (at C_n'), z_{n-1}'' (below C_{n-1}''), $z_{n-1} = \partial z_n$ (below C_{n-1})

- $z_n \mapsto z_n'$
- $\delta(z_n') = z_{n-1}''$ s.t.

$i: z_{n-1}'' \longmapsto \partial z_n = 0$

0

(b) • Suppose $\delta(\overset{z_n'}{[z_n']}) = 0$, let z_n, z_{n-1} & z_{n-1}'' be as usual,

then $[z_{n-1}''] = 0$ in $H_{n-1}'' \Rightarrow z_{n-1}'' = \partial b_n''$

- Let $\hat{z}_n = z_n - i(b_n'') \Rightarrow \partial \hat{z}_n = z_{n-1} - z_{n-1} = 0$
- $j(\hat{z}_n) = j(z_n) - j \circ i(b_n'') = z_n'$. Thus, $j_*([\hat{z}_n]) = [z_n']$.

$$H'_{n+1} \xrightarrow{\delta} H''_n \xrightarrow{i_*} H_n \longrightarrow H'_n$$

Step 4: Exactness at H''_n

$$\begin{tikzcd}
0 \arrow[r] & C''_{n+1} \arrow[r] \arrow[d] & \overset{\in z_{n+1}}{C_{n+1}} \arrow[r] \arrow[d, "\partial z_{n+1}", "z_n =" '] & \overset{\in z'_{n+1}}{C'_{n+1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z''_n \, C''_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C'_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C''_{n-1} \arrow[r] & C_{n-1} \arrow[r] & C'_{n-1} \arrow[r] & 0
\end{tikzcd}$$

(a) $i_* \circ \delta([z'_{n+1}]) = 0$; $i_* \circ \delta([z'_{n+1}]) = [i_\#(z''_n)] = [\partial z_{n+1}] = 0$

(b) Suppose $[z''_n] \in \ker(i_*)$; if $z_n = i_\#(z''_n]$, $[z_n] = 0$

· $\Rightarrow \exists\, z_{n+1}$, $\partial z_{n+1} = z_n$.

· Let $z'_{n+1} = j(z_{n+1})$; $\partial z'_{n+1} = j_\# \circ i_\#(z''_n) = 0$

· Thus, $[z'_{n+1}] \in H'_{n+1}$ & $\delta([z'_{n+1}]) = [z''_n]$

o

Addendum: **Naturality.** (should be part of main statement)

Given a commutative diagram of s.e.s.'s of chain complexes & chain homomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & C''_* \arrow[r] \arrow[d] & C_* \arrow[r] \arrow[d] & C'_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{C}''_* \arrow[r] & \hat{C}_* \arrow[r] & \hat{C}'_* \arrow[r] & 0
\end{tikzcd}$$

We get a commutative diagram of long exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H'_{n+1} \arrow[r, "\delta"] \arrow[d] & H''_n \arrow[r] \arrow[d] & H_n \arrow[r] \arrow[d] & H'_n \arrow[r, "\delta"] \arrow[d] & H''_{n-1} \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & \hat{H}'_{n+1} \arrow[r, "\delta"] & \hat{H}''_n \arrow[r] & \hat{H}_n \arrow[r] & \hat{H}'_n \arrow[r, "\delta"] & \hat{H}''_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Rk: Only the naturallity of δ is new.

28/2/2011

Lecture 12: Excision & Mayer-Vietoris.

Excision Axiom: If (X, A) is a pair of spaces, $B \subset A$ is such that $\overline{B} \subset \mathring{A}$, then the inclusion map induces isomorphisms

$$i_*: H_n(X \setminus B, A \setminus B) \xrightarrow{\cong} H_n(X, A) \quad \forall n \geqslant 0.$$

We shall prove this modulo 'small simplices lemma'

- Let X be a topological space, $\mathcal{U} = \{U_i\}_{i \in I}$ a collection of subsets of X s.t. $X = \bigcup_{i \in I} \mathring{U}_i$.
- Let $C_n^{\mathcal{U}}(X)$ be the free abelian group generated by $\{\sigma_n : \Delta^n \longrightarrow X$ a map : $\exists i \in I$ s.t. $\sigma_n(\Delta^n) \subset U_i\}$

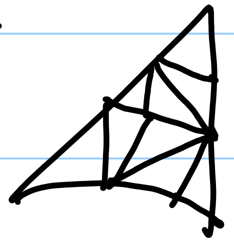
- $\partial_n|_{C_n^{\mathcal{U}}} : C_n^{\mathcal{U}} \longrightarrow C_{n-1}^{\mathcal{U}}$ $(\sigma \in C_n^{\mathcal{U}} \Rightarrow \partial\sigma \in C_{n-1}^{\mathcal{U}})$

Thus, we have a chain complex $C_*^{\mathcal{U}}(X)$, and hence associated homology groups $H_*^{\mathcal{U}}(X)$.

- $i : C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ is a chain homomorphism and hence induces homomorphisms $i_* : H_*^{\mathcal{U}}(X) \to H_*(X)$

Lemma: $i_* : H_n^{\mathcal{U}}(X) \longrightarrow H_n(X)$ is an isomorphism $\forall n \geq 0$.

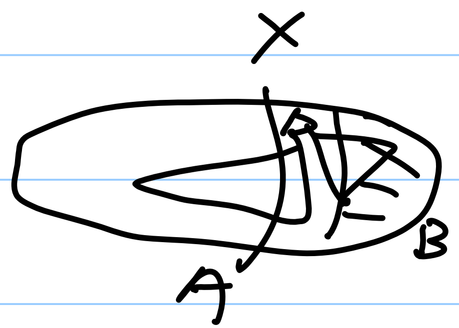
i.e., we can break a singular simplex into small pieces.



Proof later

Proof of Excision Axiom: $H_n(X \setminus B, A \setminus B) \xrightarrow{\cong} H_n(X, A)$

- Let $\mathcal{U} = \{A, X \setminus B\}$



$$\mathring{A} \cup (X \setminus \mathring{B}) = \mathring{A} \cup (X \setminus \bar{B}) = X$$

as $\bar{B} \subset \mathring{A}$.

Hence by the 'small simplices lemma',

$$H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X), \text{ induced by inclusion.}$$

Now, we have a short exact sequence

$$0 \to C_*(A) \longrightarrow C_*^{\mathcal{U}}(X) \longrightarrow C_*^{\mathcal{U}} / C_*(A) \to 0$$

- Now, $C_n^{\mathcal{U}} / C_n(A) = \dfrac{C_n(A) + C_n(X \setminus B)}{C_n(A)} \cong \dfrac{C_n(X \setminus B)}{C_n(A) \cap C_n(X \setminus B)}$

$$\begin{gathered} \rotatebox{90}{$\cong$} \\ C_n(X \setminus B) / C_n(A \setminus B) \end{gathered}$$

$[HK/H \cong K/H \cap K]$

Thus, we have short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C_*(A) \arrow[r] \arrow[d, "id"] & C_*^{\mathcal{U} = \{A, X \setminus B\}}(X) \arrow[r] \arrow[d] & \overset{C_*^{\mathcal{U}}(X)/C_*(A)}{\overset{\cong}{C_*(X \setminus B, A \setminus B)}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

forming a commutative diagram.

Hence we get a commutative diagram of long exact sequences

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d, "id_*"', "\cong"] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\cong"'] & H_n(X \setminus B, A \setminus B) \arrow[r] \arrow[d, "i_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "id_*"', "\cong"] & H_{n-1}^{\mathcal{U}}(X) \arrow[d, "\cong"] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

with $H_*^{\mathcal{U}}(X) \xrightarrow{\cong} H_*(X)$ by small simplices lemma

- Hence $i_* : H_n(X \setminus B, A \setminus B) \xrightarrow{\cong} H_n(X, A)$ by five lemma.

Exercise: Let $f : (X, A) \longrightarrow (Y, B)$ be a map of pairs of spaces (i.e. $f : X \to Y$, $f(A) \subset B$) s.t. $f : X \longrightarrow Y$ & $f_{|A} : A \to B$ are homotopy equivalences. Then

$$f_* : H_n(X, A) \xrightarrow{\cong} H_n(Y, B) \quad \forall n \geq 0.$$

Propn: If X is non-empty path connected, then $H_0(X) \cong \mathbb{Z}$.

Pf: Let $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ be defined by

$$C_0 \ni \sum_{i=1}^{k} n_i \sigma_i \mapsto \sum_{i=1}^{k} n_i \quad (\sigma_i : \Delta^0 \longrightarrow X).$$

- If $\sigma_1 \in C_1(X)$, $\sigma_1 : \overset{= \Delta^1}{[0,1]} \longrightarrow X$, $\partial \sigma_1 = \sigma_1(1) - \sigma_1(0)$

$\Rightarrow$ $\varepsilon(\partial \sigma_1) = 1 - 1 = 0.$

Hence ε induces a homomorphism

$$\tilde{\varepsilon} : \underset{\overset{\shortparallel}{C_0(X)/\partial_0 C_1(X)}}{H_0(X)} \longrightarrow \mathbb{Z}$$

· $\bar{\varepsilon}$ is onto as if $\sigma_0 : \Delta^0 \mapsto x \in X$, then

$$\bar{\varepsilon}([\sigma_0]) = 1$$

<u>Lemma:</u> $\ker(\bar{\varepsilon}) \subset \operatorname{im}(\partial_1)$

<u>Pf:</u> We proceed by induction on $\overset{=M(z)}{\sum_{i=1}^{k} |n_i|}$ for

$z = \sum_{i=1}^{k} n_i \langle v_i \rangle \in C_0(X)_{\mathbb{Z}}$, ($v_i$ distinct) Suppose $z \in \ker(\bar{\varepsilon})$, then

w.l.g. $n_1 > 0$ and $n_2 < 0$.

Then $z = (\langle v_1 \rangle - \langle v_2 \rangle) + z'$, $M(z') = M(z) - 2$

- By induction hypothesis $z' \in \mathrm{im}(\partial_1)$.
- As X is path-connected, $\exists\, \sigma : [0,1] \longrightarrow X$ with

  $\sigma(0) = v_2$ & $\sigma(1) = v_1$.

  Then $\partial\sigma = \langle v_1 \rangle - \langle v_2 \rangle$, hence $\langle v_1 \rangle - \langle v_2 \rangle \in \mathrm{im}(\partial_1)$

  $\Rightarrow z = \langle v_1 \rangle - \langle v_2 \rangle + z' \in \mathrm{im}(\partial_1)$ $\square$

Thus, $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ with $\ker(\varepsilon) = \mathrm{im}(\partial_1)$

$$\therefore \quad \bar{\varepsilon} : \underset{\substack{\shortparallel \\ H_0(X).}}{C_0(X)} / \mathrm{im}(\partial_1) \xrightarrow{\cong} \mathbb{Z}.$$

· Suppose $\{X_i\}_{i \in I}$ are the path components of a space X.

Propn: $H_n(X) = \bigoplus_{i \in I} H_n(X_i)$

Direct sums: If $\{M_i\}_{i \in I}$ are R-modules (e.g. Abelian groups, then

$$\bigoplus_{i \in I} M_i = \Big\{ \sum_{i \in I} m_i \ \big(\text{formal} \overset{\text{finite}}{\text{linear}} \text{ combinations}\big) : m_i \in M_i \Big\}$$

This satisfies: Given $\varphi_\alpha : M_i \to N$ homomorphisms,

$\exists ! \ \varphi : \bigoplus M_i \to N$ homomorphism s.t.

$$\begin{tikzcd} M_i \arrow[rr] \arrow[dr] & & N \\ & \bigoplus_i M \arrow[ur] & \end{tikzcd}$$

commutes.

More precisely: Given a collection $\{M_i\}_{i\in I}$ of R-modules, *the* direct sum of M_i is an R-module $\bigoplus_i M_i$ and a collection of (inclusion) homomorphisms $j_i : M_i \to \bigoplus_{i\in I} M_i$ s.t. given an R-module N and a collection of homomorphisms $\varphi_i : M_i \to N$, $\exists!$ homomorphism $\varphi : \bigoplus M_i \to N$ s.t.

$$\begin{tikzcd} M_i \arrow[r, "\varphi_i"] \arrow[dr, "j_i"'] & N \\ & \bigoplus M_i \arrow[u, "\varphi"'] \end{tikzcd}$$

commutes $\forall i$

Exercise: The direct sum is unique up to isomorphisms that commute with inclusions, i.e.

$$\begin{tikzcd} & M_i \arrow[dl, "j_i"'] \arrow[dr, "j_i'"] & \\ \bigoplus M_i \arrow[rr, "\cong"] & & \bigoplus' M_i \end{tikzcd}$$

commutes $\forall i$.

Direct Products:

Given $\{M_i\}_{i\in I}$ R-modules, the direct product $\prod_{i\in I} M_i$ is an R-module together with homomorphisms $\prod_{i\in I} M_i \xrightarrow{\pi_i} M_i$ s.t. given an R-module N and homomorphism $\varphi_i : N \to M_i$, $i\in I$, $\exists!$ homomorphism $\varphi: N \to \prod_{i\in I} M_i$ s.t.

$$\begin{tikzcd} N \arrow[r, "\varphi"] \arrow[dr, "\varphi_i"'] & \prod M_i \arrow[d, "\pi_i"] \\ & M_i \end{tikzcd}$$

commutes $\forall i$.

Rk: This is unique; $\prod M_i = \{ f : I \to \bigcup M_i,\ f(i) \in M_i \}$.

$\shortparallel$

$(m_1, m_2, m_3 \ldots)$

2/3/2011

Lecture 13:

Exercise: Suppose $\{M_i\}_{i\in I}$ are free R-modules with bases $\{B_i\}_{i\in I}$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Then $M = \bigoplus_{i\in I} M_i$ is the free R-module with basis $B = \bigcup_{i\in I} B_i$ (prove using universal properties)

Propn: Suppose $\{X_i\}_{i\in I}$ are the path components of a space X. Then $H_*(X) = \bigoplus_{i\in I} H_*(X_i)$

Proof: We shall show

$$C_*(X) = \bigoplus_{i\in I} C_*(X_i).$$

- $C_n(X)$ = free abelian group with basis

$$B = \{\sigma : \Delta^n \longrightarrow X \text{ map}\}$$

- $C_n(X_i)$ = free abelian group with basis

$$B_i = \{\sigma : \Delta^n \longrightarrow X_i \text{ map}\}$$

- As $\{X_i\}_{i\in I}$ are the path-components of X,

$$B = \bigcup_{i\in I} B_i. \quad \& \quad B_i \cap B_j = \emptyset \text{ for } i \neq j.$$

- Hence $C_n(X) = \bigoplus_{i\in I} C_n(X_i) \ \forall n.$
- The result for homology follows. (Exercise)

$\square$

<u>Cor:</u> $H_0(X) = \bigoplus_{\text{path-components}} \mathbb{Z}$

<u>E.g.</u> $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ $\left[S^0 = \{x \in \mathbb{R}^1 : |x| = 1\} = \{-1, 1\} \right.$

Reduced homology:

Recall that we have a homomorphism

- $\varepsilon : C_0 \longrightarrow \mathbb{Z}$ (augmentation homomorphism)

  $\sum_{i=1}^{k} n_i \langle v_i \rangle \longmapsto \sum_{i=1}^{k} n_i$

- $\varepsilon \circ \partial_1 = 0$

Hence,

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \nearrow \text{"}= C_{-1}\text{"}$$

is a chain complex. Its homology is called the <u>reduced homology</u> $\tilde{H}_*(X)$.

$$\left(\begin{array}{l} \text{Morally:} \\ \tilde{H}_{-1} = \mathbb{Z}/_{im(\varepsilon)} = \{0\} \end{array} \right)$$

- $\tilde{H}_0(X) = \frac{\ker(\varepsilon)}{im(\partial_1)}$.

Propn: · $H_n(X) = \tilde{H}_n(X) \quad \forall\, n \geq 1$

· $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$.

E.g. If X is path-connected, then

$$\tilde{H}_0(X) = \ker(\varepsilon)/_{\mathrm{im}(\partial_1)} = 0 \quad \text{(as we have seen)}$$

while $H_0(X) = \mathbb{Z}$.

Pf: $H_n(X) = \tilde{H}_n(X)$ obvious.

- $H_0(X) = C_0(X)/_{\mathrm{im}(\partial_1)}$, $\tilde{H}_0(X) = \ker(\varepsilon)/_{\mathrm{im}(\partial_1)}$
- ε induces a homomorphism

$$\bar{\varepsilon} : H_0(X) \longrightarrow \mathbb{Z}$$

with $\ker(\bar{\varepsilon}) = \tilde{H}_0(X) = \ker(\varepsilon)/_{\mathrm{im}(\partial_1)}$.

We thus have a short exact sequence

$$0 \to \tilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0.$$

We shall use the result (proved later)

Theorem: If

$$0 \to N \to M \to F \to 0$$

is a s.e.s. of R-modules with F free,

then $M = N \oplus F$ $\square$

E.g. $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a s.e.s.

but $\mathbb{Z} \neq \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

Propn: $\tilde{H}_n(S^0) = \begin{cases} \mathbb{Z}, & n=0 \\ 0 & \text{otherwise} \end{cases}$

Pf: · $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \tilde{H}_0(S^0) = \mathbb{Z}$.

· $H_n(S^0) = H_n(pt) \oplus H_n(pt) = 0 \quad \forall n \geq 1$.

We shall show:

Theorem: $\tilde{H}_n(S^k) = \begin{cases} \mathbb{Z}, & n=k \\ 0 & \text{otherwise.} \end{cases}$

We prove this inductively using the 'Mayer-Vietoris exact sequence'.

$$\begin{tikzcd} & X_1 \cap X_2 \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X_1 \arrow[dr, "j_1"'] & & {} \arrow[dl, "j_2"] \\ & X & \end{tikzcd}$$

Mayer-Vietoris: Let $X = X_1 \cup X_2$; $X_1, X_2 \subset X$ open

Then there is a long exact sequence of homology groups.

$$\cdots \to H_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, -i_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_{1*} \oplus j_{2*}} H_n(X) \xrightarrow{\partial} H_{n-1}(X_1 \cap X_2) \to \cdots$$

· In reduced homology, we have a similar long exact sequence

$$\cdots \to \tilde{H}_0(X_1 \cap X_2) \to \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2) \to \tilde{H}_0(X) \to 0$$

if $X_1 \cap X_2 \neq \emptyset$.

Proof: Let $\mathcal{U} = \{X_1, X_2\}$. Then $H_*^{\mathcal{U}}(X) = H_*(X)$ and we have a short exact sequence of chain complexes

$$0 \longrightarrow C_*(X_1 \cap X_2) \xrightarrow{(i_{1\#}, -i_{2\#})} C_*(X_1) \oplus C_2(X_2) \xrightarrow{j_{1\#} \oplus j_{2\#}\mathcal{U}} C_*(X) \longrightarrow 0$$

· This is exact as:

$$\text{if } \sigma \in C_*(X_1 \cap X_2),\ (j_{1\#} \oplus j_{2\#}) \circ (i_{1\#}, -i_{2\#})\sigma$$

$$\shortparallel$$

$$(j_{1\#} \oplus j_{2\#})(\sigma, -\sigma)$$

$$\shortparallel$$

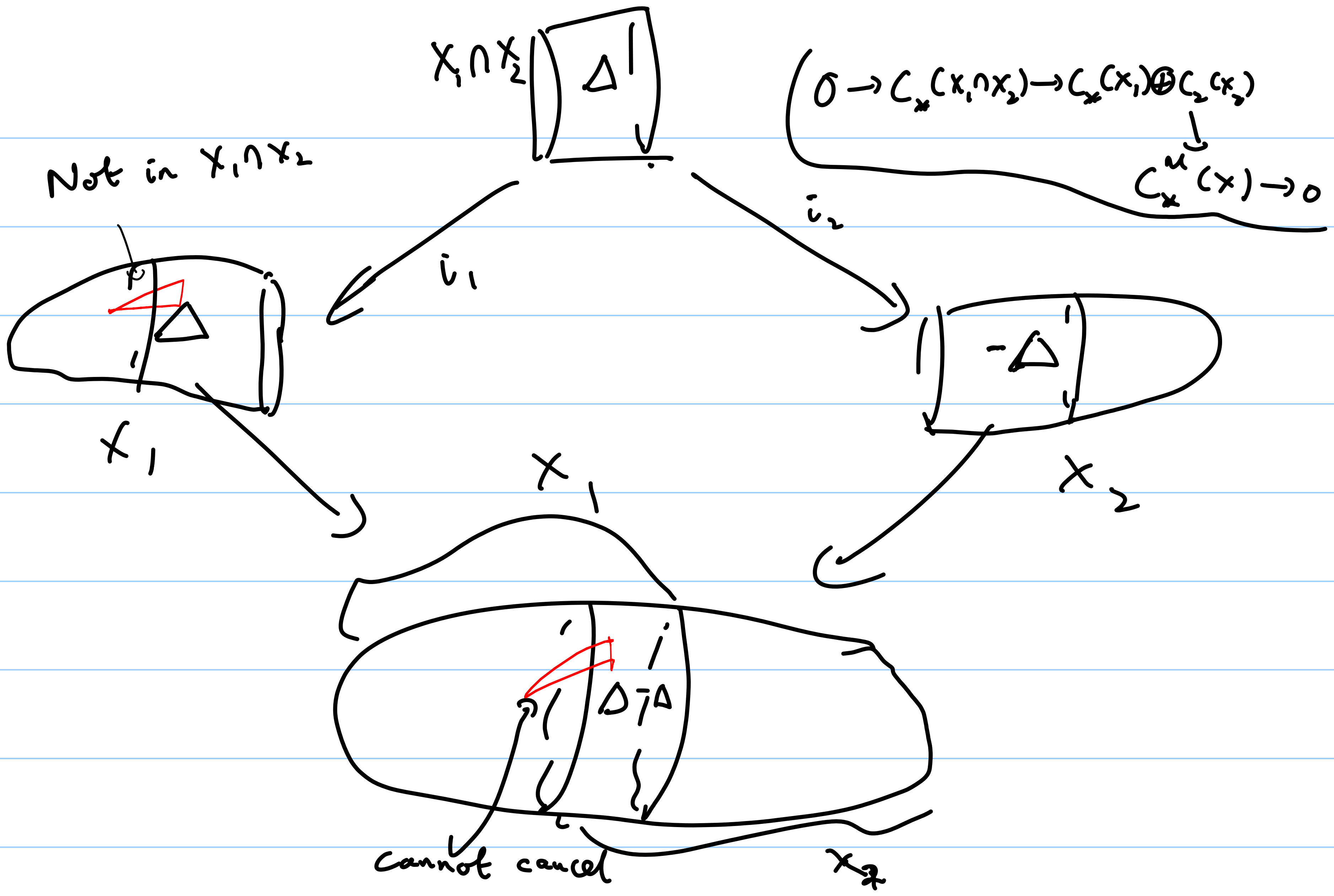
$$\sigma + (-\sigma) = 0$$

Conversely, if $\overset{\in C_n(X_1) + C_n(X_2)}{(\sum n_i \sigma_i, \sum m_i \tau_i)} \in \ker(j_{1*} \oplus j_{2*})$,

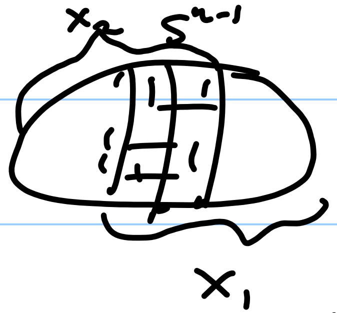
then, each $\sigma_i \in X_1 \cap X_2 \Rightarrow \sum n_i \sigma_i = i_{1\#}(\xi)$; $\sum m_i \tau_i = -i_{2\#}(\xi)$

$\triangle$

$$0 \to C_*(X_1 \cap X_2) \to C_*(X_1) \oplus C_*(X_2) \to C_*^{U}(X) \to 0$$



- This s.e.s. together with $H_*^{U}(X) = H_*(X)$ gives the Mayer-Vietoris sequence.
  - Reduced case is similar.

Homology of spheres: $S^n =$ 

- Let X_1 & X_2 be open neighbourhoods of the two hemispheres.
  - $X_1 \cap X_2$ deformation retracts to S^{n-1}, so $\tilde{H}_*(X_1 \cap X_2) = \tilde{H}_*(S^{n-1})$
  - $\tilde{H}_*(X_i) = 0$ as X_i is contractible.

Thus, the Mayer-Vietoris sequence gives

$$\tilde{H}_k(X_1) \oplus \tilde{H}_k(X_2) \longrightarrow \tilde{H}_k(X) \longrightarrow \tilde{H}_{k-1}(X_1 \cap X_2) \rightarrow \tilde{H}_{k-1}(X_1) \oplus \tilde{H}_{k-1}(X_2)$$

which for X_1, X_2 as above is

$$0 \longrightarrow \tilde{H}_k(S^n) \longrightarrow \tilde{H}_{k-1}(S^{n-1}) \longrightarrow 0$$

Thus, $\boxed{\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}), \quad k \geq 1.}$ 'Induction Equation'

Further,

$$\begin{cases} \tilde{H}_0(S^n) = 0, & n \geq 1 \\ \tilde{H}_k(S^0) = \begin{cases} \mathbb{Z}, & k = 0 \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Lemma: $\tilde{H}_k(S^n) = \begin{cases} \tilde{H}_0(S^{n-k}) & \text{if } n \geq k \\ \tilde{H}_{k-n}(S^0) & \text{if } k \geq n. \end{cases}$

Pf: By induction on $\min(n,k) \geq 0$.

- If $n=0$ or $k=0$, this is clear.
- If $n,k \geq 1$, use 'induction equation'.

Hence we prove

Theorem: $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k=n \\ 0 & \text{otherwise}. \end{cases}$

Pf: $\tilde{H}_n(S^n) = \tilde{H}_0(S^0) = \mathbb{Z}$

- If $k<n$, $\tilde{H}_k(S^n) = \tilde{H}_0(S^{n-k}) = 0$ as $n-k \geq 1$

If $k>n$, $\tilde{H}_k(S^n) = \tilde{H}_{k-n}(S^0) = 0$ as $k-n \geq 1$.

$\square$

Theorem (No retraction theorem)

There is no map $r: D^n \to S^{n-1}$ such that $r|_{S^{n-1}} = id$, i.e.,

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "id"] \arrow[dr, "i"'] & & S^{n-1} \\
& D^n \arrow[ur, "r"'] &
\end{tikzcd}$$

commutes.

Pf: If r exists, we get a corresponding commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, phantom, "\shortparallel"] & & \mathbb{Z} \arrow[d, phantom, "\shortparallel"] \\
\tilde{H}_{n-1}(S^{n-1}) \arrow[rr, "id"] \arrow[dr, "i_*"'] & & \tilde{H}_{n-1}(S^{n-1}) \\
& \tilde{H}_{n-1}(D^n) \arrow[ur, "r_*"'] \arrow[d, phantom, "\shortparallel"] & \\
& 0 &
\end{tikzcd}$$

, i.e.

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, "id"] \arrow[dr] & & \mathbb{Z} \\
& 0 \arrow[ur] &
\end{tikzcd}$$

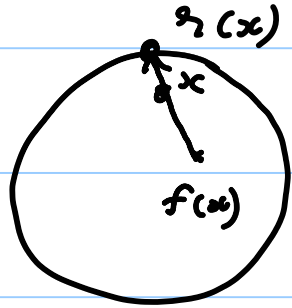
which is impossible.

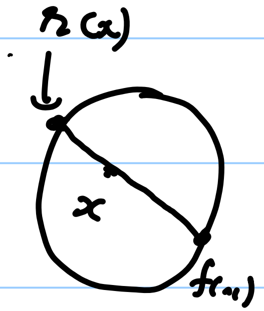
Brouwer fixed point theorem:

Any map $f: D^n \to D^n$ has a fixed point, i.e., $\exists\, P \in D^n$, $f(p) = P$.

Pf: Suppose not, we construct a retraction $r: D^n \longrightarrow S^{n-1}$ as follows:

Let $l_x(t) = (1-t) f(x) + t \cdot x$, $t \geq 0$.





- $l_x(t)$ is not constant as $f(x) \neq x$.
- Let $\tau_x = \inf \{ t \geq 1 : \| l_x(\tau) \| \geq 1 \}$ be the first point when l exist the sphere after x.
- $r(x) = l_x(\tau_x)$. [Exercise: This is continuous]

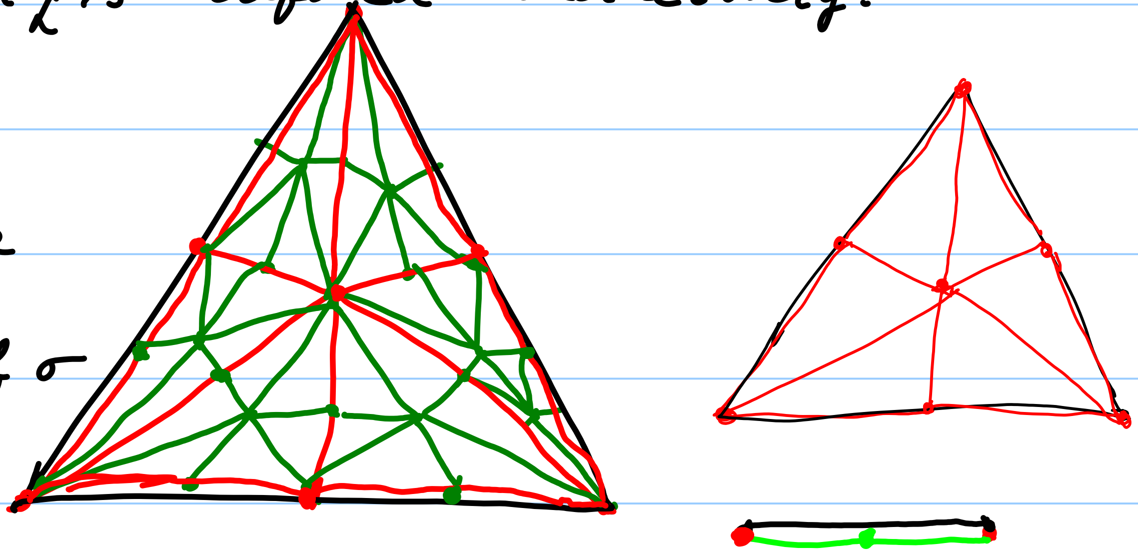
7/3/2011

Lecture 14: Barycentric Subdivision & Small simplices.

Defn: The barycenter of a simplex $\Delta^n = \langle v_0, \ldots, v_n \rangle \subseteq \mathbb{R}^N$

is $$b_\sigma = \frac{1}{n+1}\left[v_0 + \ldots + v_n\right]$$

- Barycentric subdivision S is defined inductively:
- Cone $S(\sigma)$ at b_σ.
- **Rk:** Vertices of the barycentric subdivision of σ are barycenters of subsimplices $\tau \subset \sigma$.



Combinatorial description of barycentric subdivision.

- $\Sigma = (V, S(\Sigma))$ is a simplicial complex.
- $S(\Sigma)$ is a poset (by inclusion), so we can define $br(\Sigma)$ = corresponding simplicial complex.
- A simplex in $br(\Sigma)$ is $\tau_0 \subset \tau_1 \subset \dots \subset \tau_k$, $\tau_k \in S(\Sigma)$.
- Define a map $br(\Sigma) \longrightarrow \Sigma$ linearly extending (How?)

$$\langle w_0, \dots, w_k \rangle \longmapsto \frac{1}{k+1}(w_0 + \dots + w_k)$$

$$\uparrow$$

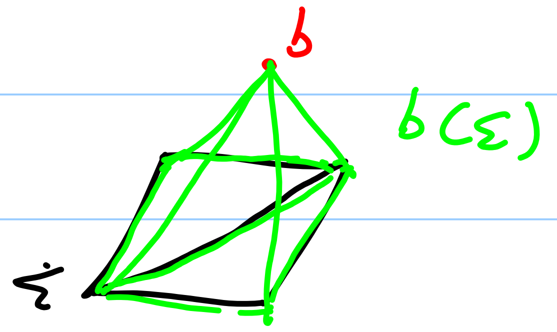
$$S(\Sigma) = V(br(\Sigma))$$

Exercise: This is a homeomorphism.

Cone: Suppose $\Sigma \subseteq \mathbb{R}^N$ is a simplicial complex and $b \in \mathbb{R}^{n+1}$ a point.

- We define $b(\Sigma)$ (cone at b of Σ) to be the simplicial complex with simplices

$$\begin{cases} \cdot\ \langle b, w_0, \dots, w_k \rangle, & \langle w_0, \dots, w_k \rangle \in S(\Sigma) \\ \cdot\ \langle w_0, \dots, w_k \rangle, & \langle w_0 \dots w_k \rangle \in S(\Sigma). \\ \cdot\ \langle b \rangle \text{ (corresponds to '}\emptyset\text{'} \end{cases}$$



Geometry: For $\sigma \subset \mathbb{R}^N$ an n-simplex the barycentric subdivision $br(\sigma)$ of σ is defined inductively: (b_σ = barycenter of σ)

- $br(pt) = pt$ (on 0-simplices)
- Given barycentric subdivisions of the faces of σ (i.e. $(n-1)$-dimensional subsimplices), the simplices of $br(\sigma)$ are

$$\left\{ \begin{array}{l} \cdot \langle b_\sigma, w_0, \dots, w_k \rangle \\ \cdot \langle w_0, \dots, w_k \rangle \end{array} \right\}, \langle w_0, \dots, w_k \rangle \in br(\tau), \tau \subset \sigma \text{ a face}$$

& $\cdot \langle b_\sigma \rangle$

Decrease of diameter:

Lemma: If $\sigma \subset \mathbb{R}^N$ is an n-simplex, then the diameter of a simplex $\hat{\sigma} \in bs(\sigma)$ satisfies

$$\operatorname{diam}(\hat{\sigma}) \leq \frac{n}{n+1} \operatorname{diam}(\sigma)$$

- If we iterate the subdivision m times, as $m \to \infty$ the diameter of a simplex in the m^{th} iterated subdivision of σ goes to 0.

Proof of lemma:

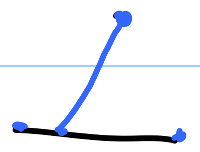
- Let $\hat{\sigma}$ be a simplex in $br(\sigma)$
- As simplices are convex, and $d(\cdot,\cdot)$ is a convex fn.

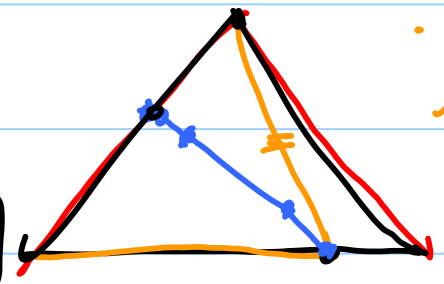
$$\mathrm{diam}(\hat{\sigma}) = \max\{d(w_i, w_j) : w_i, w_j \text{ vertices of } \hat{\sigma}\}$$

[Use: If $0 \leq a \leq 1$,

$d(x, ay + (1-a)z)$

$\leq a\,d(x,y) + (1-a)\,d(x,z)$.]





· Some orange line larger than blue line

· Some red line larger than orange line

- Any simplex $\hat{\sigma}$ is of the form:

(1) $\hat{\sigma} = \langle w_0, \dots, w_k \rangle$ in $br(\tau)$, $\tau \subset \sigma$ a face; here we use induction on n $\left(\& \ \frac{n-1}{n} < \frac{n}{n+1} \right)$

(2) $\hat{\sigma} = \langle b_\sigma, w_0, \dots, w_k \rangle$

- $d(w_i, w_j) \leq \frac{n}{n+1} \operatorname{diam}(\sigma)$ by induction
- $b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} v_i$

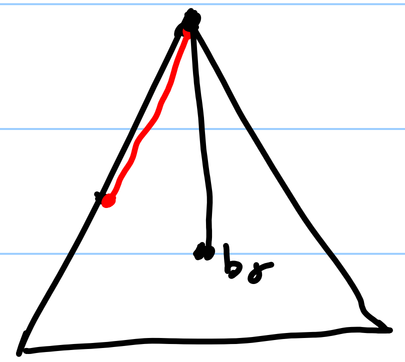
$$d(b_\sigma, w_j) = d\left(\frac{1}{n+1} \sum_{i=0}^{n} v_i, w_j \right)$$

$$\leq d\left(\frac{1}{n+1} \sum_{i=0}^{n} v_i, v_k \right) \text{ for some } k \text{ by convexity}$$

$$\leq \frac{1}{n+1} \sum_{i=0}^{n} d(v_i, v_k)$$

$$\leq \frac{1}{n+1} \cdot n \operatorname{diam}(\sigma)$$

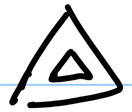
$\Big\{$ as $d(v_k, v_k) = 0$



$\square$

Algebraic Operations: (Idea: Subdivide Δ^n, $\sigma = \sigma_\#(1)$)

- Let $Z \subset \mathbb{R}^N$ be a convex set.
- $LC_n(Z) =$ linear n-chains



  $=$ free abelian group with basis

  $\{\sigma : \Delta^n \to Z,\ \sigma \text{ linear}\}$
- $LC_{-1}(Z) = \mathbb{Z}$, generated by $\emptyset$.
- $LC_0(Z) \longrightarrow \overset{\overset{\mathbb{Z}}{\shortparallel}}{LC_{-1}}(Z)$ is the augmentation homomorphism

  (i.e., $\partial_0 = \varepsilon$)
- We get a chain complex.

Cone: If $b \in Z$, define

$$b(\langle w_0, \ldots, w_k \rangle) = \langle b, w_0, \ldots, w_k \rangle$$

- This gives a homomorphism

$$LC_k \longrightarrow LC_{k+1}.$$

- $$\partial b(\langle w_0, \ldots, w_k \rangle) = \partial(\langle b, w_0, \ldots, w_k \rangle)$$
$$= \langle w_0, \ldots, w_k \rangle + \sum_{i=0}^{k} (-1)^{i+1} \langle b, \ldots \hat{w}_i, \ldots, w_k \rangle$$
$$= \langle w_0, \ldots, w_k \rangle - b(\partial \langle w_0, \ldots, w_k \rangle)$$

- Thus, $\partial b + b \partial = \mathbb{1}$.

9/3/2011

Lecture 15: Barycentric Subdivision (contd.)

Linear chains: · $Z \subset \mathbb{R}^N$ a convex set

- $\sigma : \Delta^n \longrightarrow Z$ is a linear simplex if

  $$\sigma\left(\sum_{i=0}^{n} a_i e_i\right) = \sum_{i=0}^{n} a_i \sigma(e_i), \text{ for } a_i \geq 0, \sum_{i=0}^{n} a_i = 1$$

- σ is uniquely determined by $\sigma(e_i) = w_i$.
- We denote such a linear simplex by

  $\sigma = \langle w_0, \ldots, w_n \rangle$(we may not have w_i independent.

- Linear chains $LC_n(Z) \subset C_n(Z)$ are elements of the free abelian group on linear simplices
- $\partial \langle w_0, \ldots, w_n \rangle = \sum_{i=0}^{n} (-1)^i \langle w_0, \ldots, \hat{w}_i, \ldots, w_n \rangle$ holds

- We let $LC_{-1} = \mathbb{Z}$, with basis the empty set.
- $\partial_0(\langle w_0 \rangle) = 1 \cdot \emptyset$.
- We get a chain complex

$$\dots \to LC_2 \to LC_1 \to LC_0 \to LC_{-1}$$

Cone construction: Given $b \in Z$,

define $b(\langle w_0, \dots, w_k \rangle) = \langle b, w_0, \dots, w_k \rangle$

giving a homomorphism $LC_k \longrightarrow LC_{k+1}$, $k \geq -1$.

- We saw $\partial b + b \partial = \mathbb{1}$
- For $\sigma = \langle w_0, \dots, w_n \rangle$, $b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} w_i$

Subdivision homomorphism $S : LC_* \longrightarrow LC_*$

- This is defined inductively.
- $S : LC_{-1} \longrightarrow LC_{-1}$ is the identity
- For σ a k-simplex,

$$S(\sigma) = b_\sigma (\underset{\overset{\wedge}{LC_{k-1}}}{S(\partial\sigma)}) \quad [\underline{\text{Note}}: b_\sigma \text{ depends on } \sigma]$$

Propn: $\partial S = S \partial$, i.e., S is a chain homomorphism.

Pf: We prove by induction.

$$\partial S(\sigma) = \partial b_\sigma (S(\partial\sigma)) = (1 - b_\sigma \partial)(S(\partial\sigma))$$

$$= S(\partial\sigma) - b_\sigma(\partial S(\partial\sigma))$$

Now, $b_\sigma(\partial S(\partial\sigma)) = \underline{b_\sigma(S\partial\partial\sigma)})$ by induction hypothesis

$= 0$

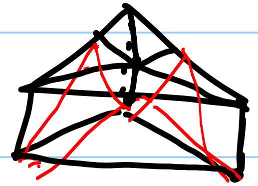
Thus, $\partial S(\sigma) = (S\partial\sigma$ ☐

- Next,

define $T : LC_k \longrightarrow LC_{k+1}$ inductively by

- $T = 0$ for $k = -1$.

- $T\sigma = b_\sigma(\sigma - T\partial\sigma)$, $k \geq 0$.



Propn: $\partial T + T\partial = \mathbb{1} - S.$

Pf: $\partial T\sigma + T\partial\sigma = \overset{\overset{1-b_\sigma\partial}{=}}{\partial b_\sigma}(\sigma - T\partial\sigma) + T\partial\sigma$

$= (\sigma - \cancel{T\partial\sigma}) - b_\sigma\partial\sigma + b_\sigma\partial T(\partial\sigma) + \cancel{T\partial\sigma}$

$$\therefore\ \partial T\sigma + T\partial\sigma = \sigma - b_\sigma(\partial\sigma) + b_\sigma(\partial T(\partial\sigma))$$

Now, proceeding inductively, we can assume

$$\partial T(\partial\sigma) + \underset{0}{\underset{\shortparallel}{T\partial(\partial\sigma)}} = \partial\sigma - S(\partial\sigma).$$

$$\therefore\ \partial T\sigma + T\partial\sigma = \sigma - \cancel{b_\sigma(\partial\sigma)} + \cancel{b_\sigma(\partial\sigma)} - b_\sigma(S(\partial\sigma))$$

$$= \sigma - S(\sigma) = (\mathbb{1} - S)(\sigma)$$

□

S & T on $C_*(X)$:

Define $S : C_*(X) \longrightarrow C_*(X)$, $T : C_*(X) \rightarrow C_{*+1}(X)$

by $S(\underset{\substack{\uparrow \\ C_k(X)}}{\sigma}) = \sigma_\#(S\underset{\substack{\uparrow \\ LC_k(\Delta^k)}}{\mathbb{1}})$, $T(\sigma) = \sigma_\#(T\mathbb{1})$

$\sigma : \Delta^k \to X$

$\shortparallel$

$\sigma_\#(\mathbb{1})$

- $S : C_*(X) \longrightarrow C_*(X)$ is a chain homomorphism
- $T : C_*(X) \longrightarrow C_{*+1}(X)$ satisfies

$$\partial T + T\partial = \mathbb{1} - S.$$

Iterated subdivision:

For $m \geq 1$, $S^m = \underbrace{S \circ S \circ \dots \circ S}_{m \text{ times}} : C_*(X) \to C_*(X)$ is a chain homomorphism.

- Let $D_m = \sum_{i=0}^{m-1} T S^i$

Propn: $\partial \circ D_m + D_m \circ \partial = \mathbb{1} - S^m$

Pf: $\partial D_m + D_m \partial = \sum_{i=0}^{m-1} (\partial T S^i + T S^i \partial) = \sum_{i=0}^{m-1} (\partial T + T\partial) S^i$

$= \sum_{i=0}^{m-1} (\mathbb{1} - S) S^i = \mathbb{1} - S^m$ $\square$

Goal: Show $C_*^{\mathcal{U}}(X)$ is chain homotopic to $C_*(X)$.

- We have $i: C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$
- We wish to construct

$$C_*(X) \xrightarrow{\rho} C_*^{\mathcal{U}}(X)$$

a chain homomorphism s.t. $\rho \circ i$ & $i \circ \rho$ are chain homotopic to the identity.

Main subtlety: We need to subdivide different simplices σ different number of times, so not consistent with ∂.

Lemma: Given $\sigma : \Delta^n \to X$ and $\mathcal{U} = \{U_i\}$, $U_i \subset X$ s.t. $\bigcup_i \mathring{U}_i = X$, $\exists\, m(\sigma)$ s.t.

$$S^{m(\sigma)}(\sigma) \in C^{\mathcal{U}}(X).$$

Pf: $\sigma^{-1}(\mathring{U}_i)$ forms an open cover of Δ^n. Hence there is a Lesbesgue number $\delta > 0$ s.t. any set of diameter $< \delta$ is contained in $\sigma^{-1}(\mathring{U}_i)$ for some i.

- As each simplex in the mth subdivision has diameter $\leq \left(\frac{n}{n+1}\right)^m$, for m sufficiently large, $S^m(\sigma) \in C^{\mathcal{U}}(X)$

$\square$

- $m(\sigma)$ depends on σ ($m(\sigma) \geq 0$ smallest integer satisfying lemma)
- If $\tau \subset \sigma$ is a subsimplex,
  - $m(\tau) \leq m(\sigma)$
  - we may have $m(\tau) < m(\sigma)$
- Define, for each k, a homomorphism

$$\hat{S} : C_k(X) \longrightarrow C_k^{u}(X)$$

$$\hat{S} : \sigma \longmapsto S^{m(\sigma)}(\sigma),$$

- Unfortunately: this is <u>not</u> a chain homomorphism.

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \tau_i, \quad \hat{S}(\partial\sigma) = \sum_{i=0}^{n} (-1)^i S^{m(\tau_i)} \tau_i$$

$$\text{but} \quad \partial \underset{S^{m(\sigma)}\sigma}{\underset{\parallel}{\hat{S}(\sigma)}} = \sum_{i=0}^{n} (-1)^i S^{m(\sigma)} \tau_i .$$

To define ρ:

Let $D(\sigma) = D_{m(\sigma)}(\sigma)$ $\left[D_{m(\sigma)} = \sum_{i=0}^{m(\sigma)-1} T S^i \right]$

- We have

$$\underset{\underset{\partial D_\sigma}{\shortparallel}}{\partial D_{m(\sigma)}}\,\sigma + \underset{\underset{D\sigma}{\nparallel \text{ in general}}}{D_{m(\sigma)}} \cdot \partial\sigma = \mathbb{1} - S^{m(\sigma)}$$

$$\therefore \partial D\sigma + D\partial\sigma = \mathbb{1} - [\underbrace{S^{m(\sigma)} + D_{m(\sigma)}\partial\sigma - D\partial\sigma}_{\rho(\sigma)}]$$

Lemma: $\rho(\sigma)$ is a chain homomorphism.

(Follows from the next lemma)

- $\partial D + D\partial = \mathbb{1} - \rho$

Lemma: If $H : C_*(x) \longrightarrow C_{*+1}(x)$ is a collection of homomorphisms, then $\varphi = \underset{\underset{[\partial, H] \text{ (notation)}}{\|}}{\partial H + H\partial}$ is a chain homomorphism.

Pf: $\varphi \circ \partial = \partial H \partial + H \partial\partial = \partial H \partial$

$\partial \varphi = \partial\partial H + \partial H \partial = \partial H \partial$ $\square$

Lemma: $\rho(\sigma) = S^{m(\sigma)} + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \in C^{u}(x)$ $\forall \sigma$.

Pf: $S^{m(\sigma)} \in C^{u}(x)$

- If τ is a face of σ, we show

$$D_{m(\sigma)}(\tau) - D(\tau) \in C^{u}(x)$$

- Recall $m(\tau) \leq m(\sigma)$

- $D_{m(\sigma)}\tau - \underset{D_{m(\tau)}\tau}{\overset{}{\underset{\shortparallel}{D\tau}}} = \sum_{i=0}^{m(\sigma)-1} TS^i(\tau) - \sum_{i=0}^{m(\tau)-1} TS^i(\tau)$

$$= \sum_{i=m(\tau)}^{m(\sigma)-1} TS^i(\tau)$$

- If $i \geq m(\tau)$, $S^i(\tau) \in C^u(X)$

$$\Rightarrow TS^i(\tau) \in C^u(X)$$

Thus, $D_{m(\sigma)}\tau - D\tau \in C^u(X)$ $\square$

Exercise: If $\sigma \in C^u(X)$, $\rho(\sigma) = \sigma$,

i.e. $\rho \circ i = \mathbb{1}$.

We now have

$$C_*^{\mathcal{U}}(X) \underset{p}{\overset{i}{\rightleftarrows}} C_*(X)$$, chain homomorphisms

- $p \circ i = \mathbb{1}$
- $i \circ p = p$ is chain homotopic to $\mathbb{1}$ as

$$\partial D + D \partial = \mathbb{1} - p$$

- Thus, $C_*^{\mathcal{U}}(X)$ is chain homotopic to $C_*(X)$
- This completes the verification of the main axioms for homology

☺

14/3/2011

Lecture 16: Some Applications

Already seen:

- $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0 & \text{otherwise} \end{cases}$

- No retraction theorem

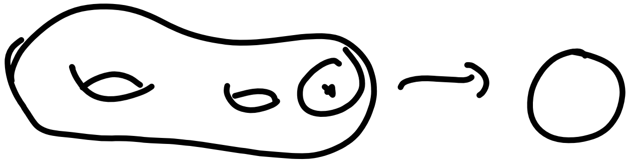
- Brouwer fixed point theorem.

Theorem: If there is a homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$, then $n = m$.

Pf: Given such an f, we get

$$\tilde{H}_*(S^{n-1}) = \tilde{H}_*(\mathbb{R}^n \setminus \{0\}) = \tilde{H}_*(\mathbb{R}^m \setminus f(0)) = \tilde{H}_*(S^{m-1})$$

$\Rightarrow m = n$ $\square$



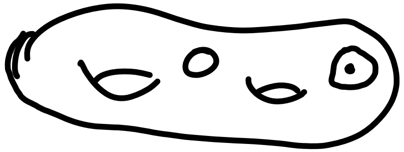
Defn: A space M is a topological n-dimensional manifold if $\forall p \in M$, $\exists\, U \subset M$, $p \in U$ and a homeomorphism $f : U \longrightarrow \mathbb{R}^n$.

Theorem: (Invariance of dimension) If $f : M_1 \longrightarrow M_2$ is a homeomorphism from a topological n_1-dimensional manifold to a topological n_2-dimensional manifold, and $M_1 \neq \emptyset$, then $n_1 = n_2$.

Pf: We shall use <u>local homology</u>.

Defn: If X is a space & $x \in X$, the local homology of X at x is $H_*(X, X \setminus \{x\})$.

(Assume X is T_1)



Propn: If $U \subset X$ is open and $x \in U$, then

$$H_*(X, X \setminus \{x\}) \cong H_*(U, U \setminus \{x\})$$

with the isomorphism induced by inclusion

Pf: We excise the closed set $X \setminus U$ contained in the open set $X \setminus \{x\}$. $\square$

Propn: If M is an n-manifold, then for $p \in M$,

$$H_k(M, M \setminus \{p\}) = \begin{cases} \mathbb{Z}, & k = n. \\ 0 & \text{otherwise} \end{cases}$$

Rk: Relative homology is the same for reduced and absolute homology. ($H_*(X, A) \cong \tilde{H}_*(X,A)$]

Pf: Let U be an open set in M containing p with U homeomorphic to $\mathbb{R}^n$.

Then $H_k(M, M\setminus\{p\}) = H_k(U, U\setminus\{p\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$
$$\underset{\tilde{H}_k(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})}{\shortparallel}$$

- Now, by the long-exact sequence in homology,

$$\underset{0}{\underset{\shortparallel}{\tilde{H}_k(\mathbb{R}^n)}} \longrightarrow \tilde{H}_k(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \longrightarrow \tilde{H}_{k-1}(\mathbb{R}^n\setminus\{0\}) \longrightarrow \underset{0}{\underset{\shortparallel}{\tilde{H}_{k-1}(\mathbb{R}^n)}}$$

$\Rightarrow$ $\tilde{H}_k(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) = \tilde{H}_{k-1}(\mathbb{R}^n\setminus\{0\}) = \hat{H}_{k-1}(S^{n-1})$

$$= \begin{cases} \mathbb{Z}, & k=n \\ 0 & \text{otherwise.} \end{cases}$$

Proof of invariance of dimension

Suppose $f: M_1 \to M_2$ is a homeomorphism

and $f(p_1) = p_2$, $p_i \in M_i$,

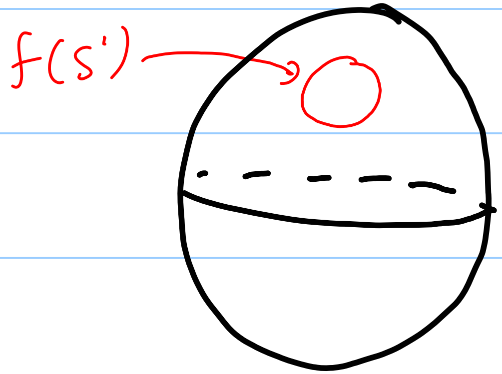
then $$\mathbb{Z} = H_{n_1}(M_1, M_1 \setminus \{p_1\}) = H_{n_1}(M_2, M_2 \setminus \{p_2\}) = \begin{cases} \mathbb{Z} & \text{if } n_1 = n_2 \\ 0 & \text{otherwise} \end{cases}$$

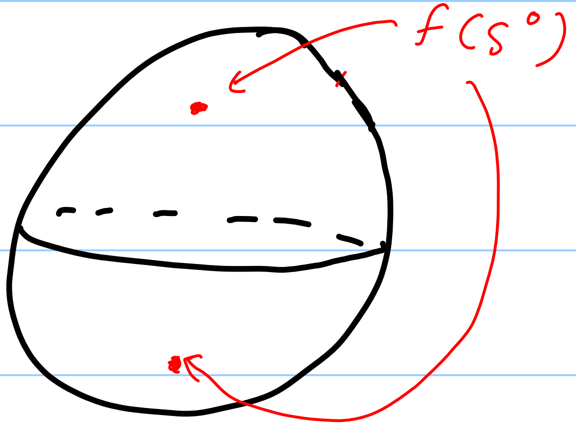
Thus, $n_1 = n_2$. $\square$

Jordan–Brouwer separation theorem: Let $f: S^{n-1} \to S^n$ be an injective map. Then $S^n \setminus f(S^{n-1})$ has two (path) components.

· We formulate this as a homological statement:

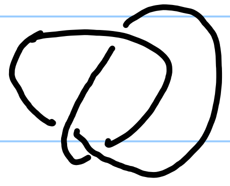
<u>Conclusion:</u> $\tilde{H}_0(S^n \setminus f(S^{n-1})) = \mathbb{Z}$.





<u>In E.g.</u> $S^2 \setminus f(S^1) \underset{h.e.}{\simeq} S^0$ <u>In e.g.</u> $S^2 \setminus f(S^0) \underset{h.e.}{\simeq} S^1$

·The analogue is false in general $S^3 \setminus K =$



but true for <u>homology</u>.

?

Alexander Duality: Let $f: S^k \longrightarrow S^n$ be an injective map. Then $\tilde{H}_m(S^n \setminus f(S^k)) = \begin{cases} \mathbb{Z}, & m=n-k-1 \\ 0 & \text{otherwise} \end{cases}$

We deduce this from:

Lemma: Let $f: \overset{=D^k}{I^k} \longrightarrow S^n$ be an injective map. Then
$$\tilde{H}_m(S^n \setminus f(I^k)) = 0 \quad \forall k.$$

Proof of duality theorem:

- Note $S^k = D^k_+ \cup D^k_-$, $D^k_\pm$ closed hemispheres,
- $D^k_+ \cap D^k_- = S^{k-1}$.
- $S^{-1} \subset D^0$ is $S^{-1} = \emptyset$.

- For $k=-1$, the duality theorem says

$$\tilde{H}_m(S^n) = \begin{cases} \mathbb{Z}, & m = n-(-1)-1 = n \\ 0 & \text{otherwise} \end{cases}$$

which is true.

- We proceed by induction on k.
- Note that $\overset{\subset\, U_\pm}{S^n \setminus f(D^k_\pm)}$ are open sets.

$$\left.\begin{array}{l} \cdot\ U_+ \cap U_- = S^n \setminus f(S^k) \\ \cdot\ U_+ \cup U_- = S^n \setminus f(S^{k-1}). \end{array}\right\} \text{ We use the Mayer-Vietoris exact sequence}$$

$$\to \overbrace{\underset{\substack{\| \\ \tilde{H}_m(S^n \setminus f(D^k_+)) \cong 0 \text{ (by lemma)}}}{\tilde{H}_m(U_+)} \oplus \tilde{H}_m(U_-)}^{=0} \to \tilde{H}_m(S^n \setminus f(S^{k-1})) \to \tilde{H}_{m-1}(S^n \setminus f(S^k)) \to \overbrace{\tilde{H}_{m-1}(U_+) \oplus \hat{H}_{m-1}(U_-)}^{=0}$$

Thus,

$$\tilde{H}_m(S^n \setminus f(S^{k-1})) = \tilde{H}_{m-1}(S^n \setminus f(S^k)) \ \forall m$$

- We now prove by induction.

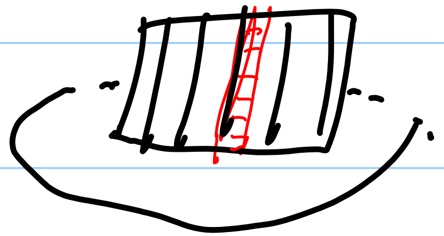
$$\tilde{H}_m(S^n \setminus f(S^k)) = \tilde{H}_{m+1}(S^n \setminus f(S^{k-1})) = \begin{cases} \mathbb{Z}, & \overset{m = n-k-1.}{\overset{\Updownarrow}{m+1 = n-(k-1)-1}} \\ 0 & \text{otherwise.} \end{cases}$$

<u>Remains to prove:</u>

Lemma: $f : I^k \to S^n$ injective map $\Rightarrow \tilde{H}_m(S^n \setminus f(I^k)) = 0 \ \forall m.$

Pf: We proceed inductively.

- For $k=0$, $S^n \setminus f(I^0) = \mathbb{R}^n$, which gives the claim.
- Assume the result for $k-1$.



- We are given

$$f : I^k \longrightarrow S^n$$

- By induction hypothesis, for $t \in I$,

$$\tilde{H}_*(S^n \setminus f(I^{k-1} \times \{t\})) \cong 0.$$

- Let $[z] \in \tilde{H}_m(S^n \setminus f(I^k))$, $z \in C_m(S^n \setminus f(I^k))$ a cycle.

- As $\tilde{H}_*(S^n \setminus f(I^{k-1} \times \{t\}) = 0$, $\forall t \in I$, $\exists\, b_t \in C_m(S^n \setminus f(I^{k-1} \times \{t\})$ s.t. $\partial b_t = z$.

- As the image of a singular chain is compact, $\exists\, U_t \subset [0,1]$ open, $t \in U_t$ s.t. $b_t \in C_{m+1}(S^n \setminus f(I^{k-1} \times U_t))$

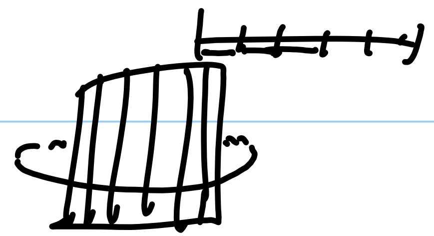
· Thus, $\exists$ open cover $\{U_t\}$ of $[0,1]$ s.t. $z = \partial b_t$

where $b_t \in C_{m+1}(S^n \setminus f(I^{k-1} \times U_t))$

i.e. $[z] = 0$ in $\tilde{H}_m(S^k \setminus f(I^{k-1} \times U_t))$

· Hence $I = I_1 \cup I_2 \cup \dots \cup I_\ell$ s.t.

$$I_p \cap I_q = \begin{cases} pt & \text{if } |p-q| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$



and $[z] = 0$ in $\tilde{H}_m(S^k \setminus f(I^{k-1} \times I_p))$ $\forall p$.

<u>Claim</u>: If $[z] = 0$ in $\tilde{H}_m(S^k \setminus f(I^{k-1} \times J_i))$, $i = 1, 2$

with J_i intervals intersecting in a point,

then $[z] = 0$ in $\tilde{H}_m(S^k \setminus f(I^{k-1} \times (J_1 \cup J_2)))$

Claim $\Rightarrow$ lemma is easy.

Claim by Mayer-Vietoris $\quad U_i = S^n \setminus f(I^{k-1} \times J_i)$

$$\underset{\overset{\shortparallel}{0}}{\tilde{H}_m(U_1) \oplus \tilde{H}_m(U_2)} \longrightarrow \underbrace{\tilde{H}_m(\underset{\overset{\shortparallel}{U_1 \cup U_2}}{S^n \setminus f(I^{k-1}} \times \underset{\overset{\shortparallel}{\mathrm{pt.}}}{(J_1 \cap J_2)}))}_{\overset{\shortparallel}{0} \text{ by induction hypothesis}} \longrightarrow \tilde{H}_{m-1}(S^n \setminus f(I^{k-1} \times (J_1 \cup J_2)) \longrightarrow 0$$

$$\Rightarrow \tilde{H}_{m-1}(S^n \setminus f(I^{k-1} \times (J_1 \cup J_2)) = 0. \qquad D.$$

16/3/2011

Lecture 17: More applications & Examples

Recall:

Theorem (Special case of Alexander duality):

$f : S^k \longrightarrow S^n$ is an injective map. Then

$$\tilde{H}_m (S^n \setminus f(S^k)) = \begin{cases} \mathbb{Z}, & m = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$

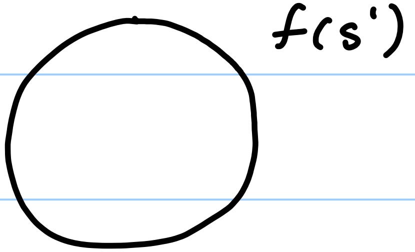
E.g. $f : S^1 \longrightarrow S^3$ is an injective map, $H_1 (S^3 \setminus f(S^1)) = \mathbb{Z}$

- However, for two such maps f & g, we may not have

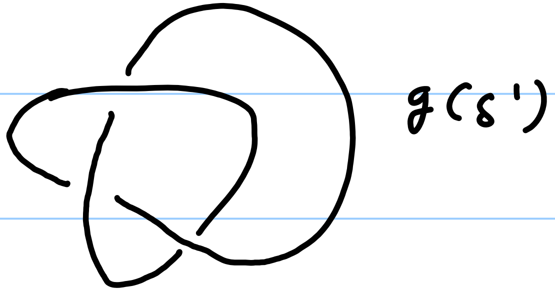
$$S^3 \setminus f(S^1) \underset{\text{h.e.}}{\sim} S^3 \setminus g(S^1)$$

as $\pi_1 (S^3 \setminus f(S^1)) \neq \pi_1 (S^3 \setminus g(S^1))$ in general.

<u>E.g.</u>



versus

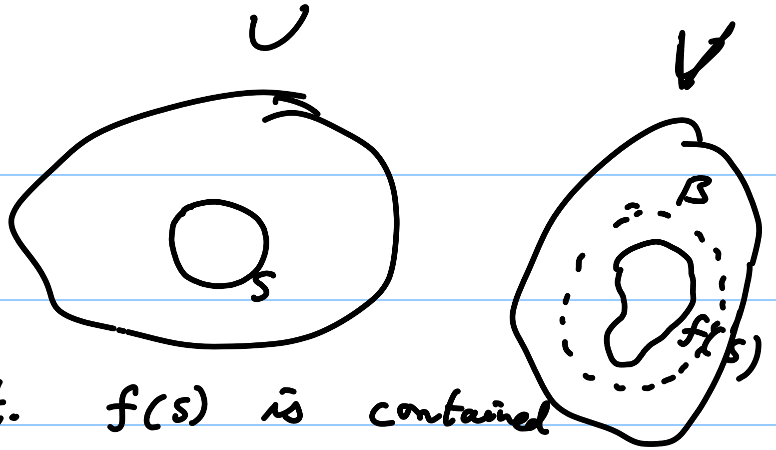


- Even for embeddings $f: S^2 \to S^3$, we may have $\pi_1(S^3 \setminus f(S^2)) \neq 1$, e.g., Alexander horned sphere.

<u>Some applications:</u>

<u>Another proof of invariance of dimension:</u>

<u>Thm</u>: If $n < m$, there is no homeomorphism $f: U \to V$ between open sets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$.



<u>Pf</u>: W.l.o.g. U and V are connected. Let $S \subset U$ be a $(n-1)$-dimensional sphere s.t. $f(S)$ is contained in a ball $B \subset V$.

- Now $f_{|U \setminus S}$ gives a homeomorphism from $U \setminus S$ to $V \setminus f(S)$. However, we show

  $\left\{ \begin{array}{l} \cdot\ U \setminus S \text{ is not connected} \\ \cdot\ V \setminus f(S) \text{ is connected} \end{array} \right.$

  giving a contradiction.

- $\mathbb{R}^n \setminus S$ is not connected by Jordan–Brouwer, from which we deduce that $U \setminus S$ is not connected.

For, $\mathbb{R}^n \setminus S = V_1 \cup V_2$, V_1 & V_2 open.

If $U \setminus S$ is connected, as

$$U \setminus S = (V_1 \cap U) \cup (V_2 \cap U),$$

at least one of $V_1 \cap U$ and $V_2 \cap U$ is empty, i.e., w.l.g. $V_2 \cap U = \emptyset$.

- Let $u \in U$ be a point and $v \in V_2$.

Consider the linear map $l : [0,1] \longrightarrow \mathbb{R}^n$,

$$l(t) = (1-t)v + tu.$$

- Consider the component α of $l \cap V_2$ containing $v = l(0)$

- If $\alpha = [0,1]$, then $u \in V_2$, contradicting $U \cap V_2 = \emptyset$
- Otherwise $\alpha = [0, \tau)$ for some τ. By definition of V_2, $l(\tau) \in S$. Hence for $\varepsilon > 0$ small enough, $l(\tau - \varepsilon) \in U$, But $l(\tau-\varepsilon) \in V_2$, contradicting $U \cap V_2 = \emptyset$.

Thus, $U \setminus S$ is not connected.

- $f(S) \subset B \subset V$, and $B \cong \mathbb{R}^m$. Hence

  $B \setminus f(S)$ is connected as $\tilde{H}_0(B \setminus f(S)) = 0$

  By a similar argument to the above, $V \setminus f(S)$ is connected.

$\square$

Review pf. of Alexander duality:

Lemma: $f : I^k \overset{1\text{-}1}{\hookrightarrow} S^n$ then $\tilde{H}_*(S^n \setminus f(I^k)) = 0$.

Deduce: $H_*(S^n \setminus f(S^k)) = \dots$ if f injective.

Mayer-Vietoris gives.

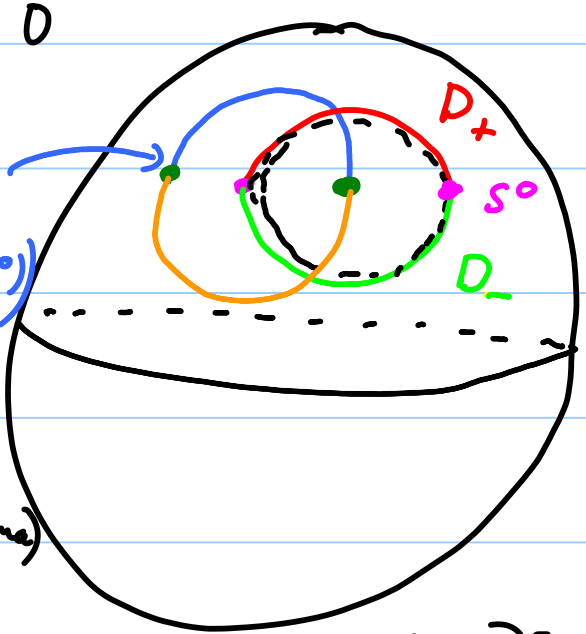
$$0 \to \tilde{H}_1(S^2 \setminus f(S^0)) \xrightarrow[\partial]{\sim} \tilde{H}_0(S^2 \setminus f(S^1)) \to 0$$

· **The map:**

A cycle $z \in C_1(S^2 \setminus f(S^0))$

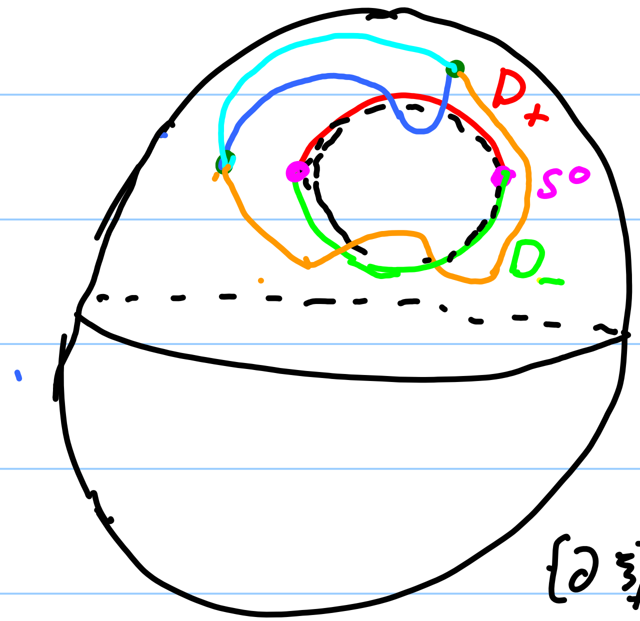
can be decomposed as $z = \xi_+ - \xi_-$,

$\xi_\pm \in C_1(S^2 \setminus D_\pm)$ (small simplices lemma)



· $\partial z = 0 \iff \partial \xi_+ = \partial \xi_-$. In Mayer-Vietoris: $\partial([z]) = \partial \xi_+ = \partial \xi_-$

$$0 \to \tilde{H}_1(S^2 \setminus f(S^0)) \xrightarrow{\partial} \tilde{H}_{\underline{0}}(S^2 \setminus f(S^1)) \to 0$$



<u>Injectivity of ∂:</u>

Suppose $[z] \mapsto 0$,

$z = \xi_+ - \xi_-$, $[\partial \xi_+] = 0$

· Assume ξ_+, ξ_- intervals

$[\partial \xi_+] = [\partial \xi_-] = [p - q] = 0$ means

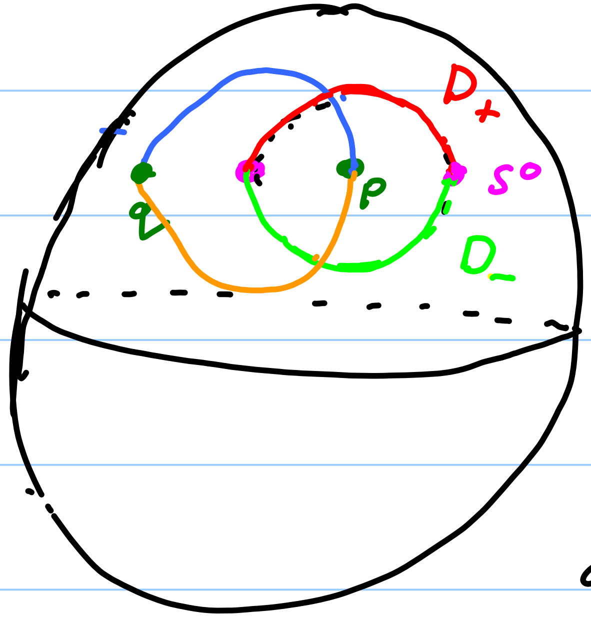
there is a path α joining p to q in $S^2 \setminus f(S^1)$

· $z = \xi_+ - \xi_- = (\underset{S^2 \setminus f(D_+)}{\underset{\uparrow}{\xi_+ + \alpha}}) - (\underset{S^2 \setminus f(D_-)}{\underset{\uparrow}{\xi_- + \alpha}})$.

But $\partial(\xi_\pm + \alpha) = 0$, so $[\xi_\pm + \alpha] \in \tilde{H}_1(S^2 \setminus f(D_\pm)) = 0$

$\implies [z] = 0$

Surjectivity: $\tilde{H}_1(S^2 \setminus f(S^0)) \xrightarrow{\partial} \tilde{H}_0(S^2 \setminus f(S^1)) \to 0$



- Let $[p-q] \in \tilde{H}_0(S^2 \setminus f(S^1))$
  We have
- $\iota[p-q] \in \tilde{H}_0(S^2 \setminus f(D_\pm)) = 0$
- Hence there are paths

$\xi_\pm \in C_1(S^2 \setminus f(D_\pm))$ with

$\partial \xi_\pm = p - q$

$z = \xi_+ + \xi_-$ is a cycle s.t.

$\partial[z] = [p-q]$

Lecture 18: Degree of a map

$f: S^n \longrightarrow S^n$ a map

Then $$\begin{tikzcd} f_*: H_n(S^n) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(S^n) \arrow[d, phantom, "\shortparallel"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Thus, $\exists\, k \in \mathbb{Z}$ s.t. $f_*([z]) = k[z] \quad \forall [z] \in H_n(S^n)$

- k is called the degree of f (denoted $\deg(f)$)

Propn: If $f: S^n \to S^n$ is a map with $\deg(f) \neq 0$, then f is surjective.

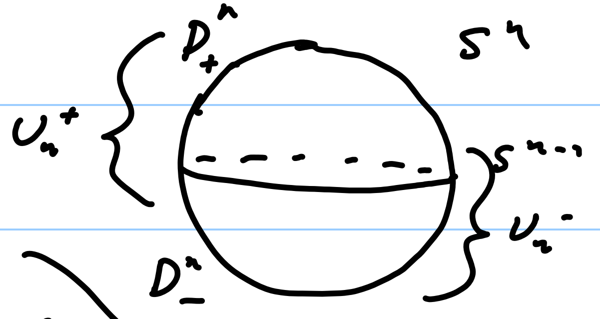
Pf: Suppose f is not onto, then $\exists\, p \in S^n$, $f(S^n) \subset S^n \setminus \{p\}$

Hence we have
$$\begin{tikzcd} H_n(S^n) \arrow[r, "f_*"] \arrow[rd, "f_*"'] & H_n(S^n) \\ & H_n(S^n \setminus \{p\}) = 0 \arrow[u, "i_*"'] \end{tikzcd} \Rightarrow f_* = 0$$

□

Propn: If $f \sim g$, then $\deg(f) = \deg(g)$ $\square$

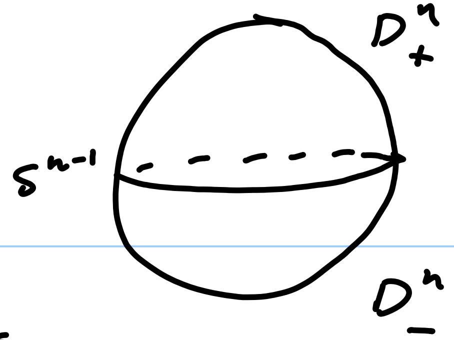
Lemma: Let $\rho : S^n \to S^n$ be a reflection fixing all points of $S^{n-1} \subset S^n$.
Then $\deg(\rho) = -1$.



Pf: Consider neighbourhoods U_n^+ & U_n^- of the hemispheres obtained by splitting along S^{n-1}.
Then $A = U_+^n \cap U_-^n$ deformation retracts to S^{n-1}, giving the Mayer-Vietoris sequence

$$\to \underset{\overset{\shortparallel}{0}}{H_n(U_+^n) \oplus H_n(U_-^n)} \to H_n(S^n) \to H_{n-1}(A) \to \underset{\overset{\shortparallel}{0}}{H_{n-1}(U_+^n) \oplus H_{n-1}(U_-^n)} \to$$

This given



$$0 \longrightarrow \underset{\mathbb{Z}}{\underset{\shortparallel}{H_n(S^n)}} \xrightarrow{\partial_*} \underset{\mathbb{Z}}{\underset{\shortparallel}{H_{n-1}(S^{n-1})}} \longrightarrow 0,$$

i.e. ∂_* is an isomorphism.

- Let $[\zeta] \in H_{n-1}(S^{n-1})$ be a generator; $\zeta \in \underset{C_{n-1}(D^n_\pm)}{\underset{\cap}{C_{n-1}(S^{n-1})}}$
- $\zeta \in C_{n-1}(D^n_+)$ gives $[\zeta] \in H_{n-1}(D^n_+) \cong 0$

Hence $\exists\ b_+ \in C_n(D^n_+)$ s.t. $\partial b_+ = \zeta$

- Under the natural identification of D^n_+ & D^n_-, we have

a chain $b_- \in C_n(D^n_-)$; $\partial b_- = \zeta$

- $\partial(b_+ - b_-) = 0$ & $\partial_*([b_+ - b_-]) = [\zeta]$.

Conclusion: $[b_+ - b_-] \in H_n(S^n)$ is the generator.

· Now, $\rho_\#(b_\pm) = b_\mp$, $\rho_\# : C_n(D^n_\pm) \to C_n(D^n_\mp)$.

Thus $\rho_\#(b_+ - b_-) = -(b_+ - b_-)$

$\therefore \rho_*([b_+ - b_-]) = -[b_+ - b_-] \in H_n(S^n)$

Hence $\deg(\rho) = -1$ $\square$

Theorem: Let $T: S^n \to S^n$ be the antipodal map

$$T(x) = -x.$$

Then $\deg(T) = (-1)^{n+1}$

Pf: Let $\rho_i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be $(x_1, \dots, x_n) \mapsto (x_1, \dots, -x_i, \dots, x_n)$

Then $T = \rho_1 \circ \rho_2 \circ \rho_3 \circ \dots \circ \rho_{n+1}$

Hence $T_* = (\rho_1)_* \circ (\rho_2)_* \circ \dots \circ (\rho_{n+1})_* = (-1)^{n+1}$ $\square$

Exercise: If $f, g : S^n \to S^n$ are maps, then

$$\deg(f \circ g) = \deg(f) \cdot \deg(g).$$ $\square$

Theorem: The antipodal map $T : S^n \to S^n$ is homotopic to the identity iff n is odd.

Pf: If $T \sim id$, $\deg(T) = \deg(1) = 1 \Rightarrow (-1)^{n+1} = 1 \Rightarrow n$ odd.

Conversely, suppose n is odd, we show $T \sim I$

Namely, if $n=2k-1$ is odd,

$$\mathbb{R}^{n+1} = \mathbb{R}^{2k} = \mathbb{C}^k$$

Define a homotopy from id to T by

$$H: S^n \times I \to S^n$$

$$H((z_1, .., z_k), t) = (e^{i\pi t}z_1, ..., e^{i\pi t}z_k) \qquad \square$$

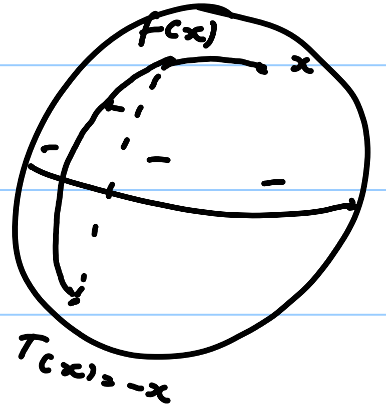
Theorem: If n is even and $f: S^n \to S^n$ is a map homotopic to id, then f has a fixed point.

Pf: If f has no fixed points, then we show $f \sim T$. But $T \not\sim \text{id}$, contradicting $f \sim \text{id}$.

Namely, we define

$$H : S^n \times I \longrightarrow S^n$$

by $H(x,t) = \dfrac{(1-t)f(x) + t\cdot(-x)}{\|(1-t)f(x) + t(-x)\|}$



As $f(x) \neq x \ \forall x$, $\underset{\displaystyle (1-t)f(x) \neq t\cdot x}{\overset{\Updownarrow}{(1-t)f(x) + t\cdot(-x)}} \neq 0 \ \forall x, t.$

<u>Theorem:</u> (Hairy ball theorem)

Let V be a continuous vector field on $S^n \subseteq \mathbb{R}^{n+1}$, n even, i.e., $V : S^n \longrightarrow \mathbb{R}^{n+1}$ is a map s.t. $\underset{\displaystyle x \perp V(x)}{\overset{\Updownarrow}{\langle x, V(x)\rangle}} = 0 \ \forall x.$

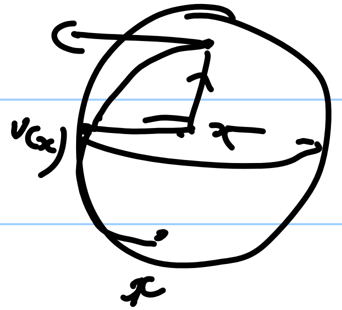
Then $\exists x \in S^n$ s.t. $V(x) = 0$

<u>**Rk:**</u> We have used in the statement

$$T_x S^n = \{ v \in \mathbb{R}^{n+1} ; \langle x, v \rangle = 0 \}$$

<u>**Proof:**</u> If $V(x) \neq 0$ for all $x \in S^{n-1}$, then w.l.g. $\|V(x)\| = 1$ $\forall x$ (replace $V(x)$ by $\frac{V(x)}{\|x\|}$)

We construct a homotopy from the identity to T, namely



$$H(x,t) = \cos(\pi t)x \underset{\substack{\uparrow \\ S^n}}{+} \sin(\pi t)V(x).$$

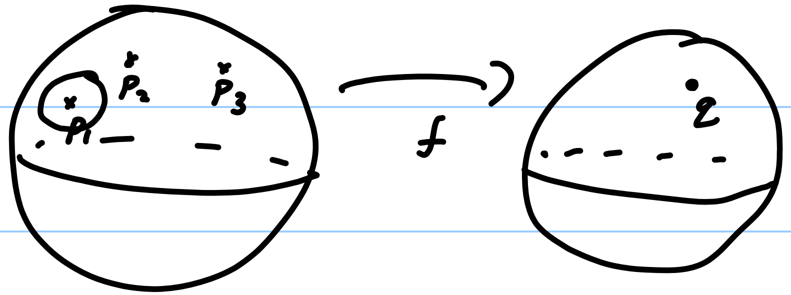
as $\|x\| = \|V(x)\| = 1$, $\langle x, v \rangle = 0$

$\square$

Local computation of degree : $\deg(f) =$ '$\# f^{-1}(q)$' with sign & multiplicity.

- Let $q \in S^n$, assume

  $f^{-1}(q) = \{p_1, \dots, p_k\}$ is finite.



- **Index at $p_i \in f^{-1}(q)$.**

  There is an open set U s.t. $f^{-1}(q) \cap U = \{p_i\}$.

Then $f : (U, U \setminus \{p_i\}) \longrightarrow (S^n, S^n \setminus \{q\})$

This gives

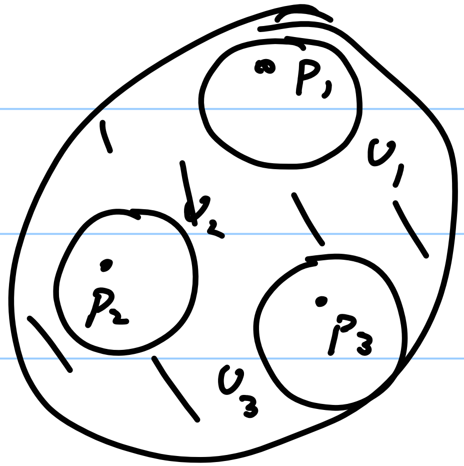
$$\begin{tikzcd} H_n(U, U \setminus \{p_i\}) \arrow[r, "f_*"] \arrow[d, "\cong"] & H_n(S^n, S^n \setminus \{q\}) = H_n(S^n) \arrow[d, phantom, "\parallel"] \\ \mathbb{Z} = H_n(S^n) \cong H_n(S^n, S^n \setminus \{p_i\}) & \mathbb{Z} \end{tikzcd}$$

Hence $f_* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by a number $\mathrm{ind}_f(p_i)$.

Theorem: $\deg(f) = \sum_{p_i \in f^{-1}(q)} \operatorname{ind}_f(p_i)$

Pf - see Hatcher

$$\begin{tikzcd}
H_n(S^n, S^n \setminus \{p_1, \dots, p_k\}) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{q\}) \\
\bigoplus H_n(U_i, U_i \setminus \{p_i\}) \arrow[u, "\cong"'] &
\end{tikzcd}$$



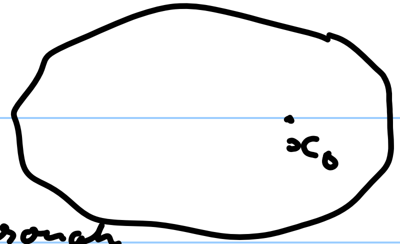
$f^{-1}(V) = U_1 \cup \dots \cup U_k$

23/3/11

Lecture 19

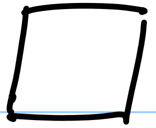
Homotopy groups π_n :

- (X, x_0) is a pointed space
- $\pi_n(X, x_0) = \{ f : I^n \to X_{x_0} \overset{\text{map}}{,} f(\partial I^n) = x_0 \}/\sim$



where equivalence is homotopy through such maps, i.e.

$f \sim g$ if $\exists H : I^n \times I \longrightarrow X$



s.t. $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, $x \in I^n$

& $H(y, t) = x_0$ if $y \in \partial I^n$, $t \in [0,1]$

Rk: We can view these as maps

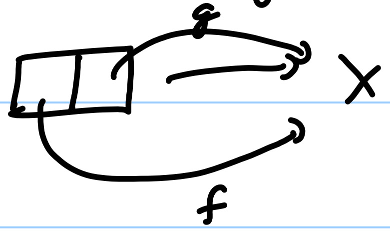
$f : (S^n, *) = I_n/_{\partial I_n} \longrightarrow (X, x_0)$

Group operation:

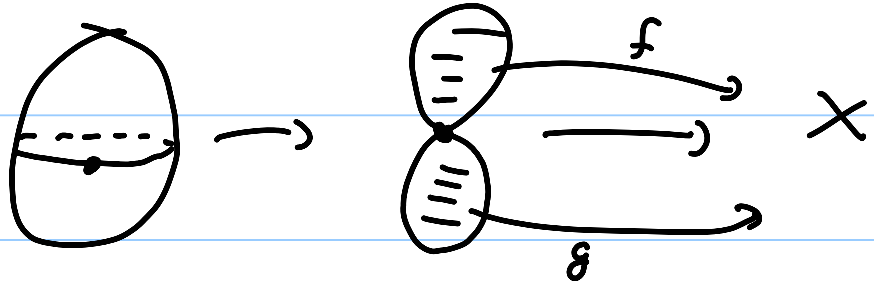
$$\boxed{f} * \boxed{g} = \boxed{f \mid g}$$

i.e., $f: \square \longrightarrow X$, $g: \square \to X$

$f * g$:



In terms of spheres

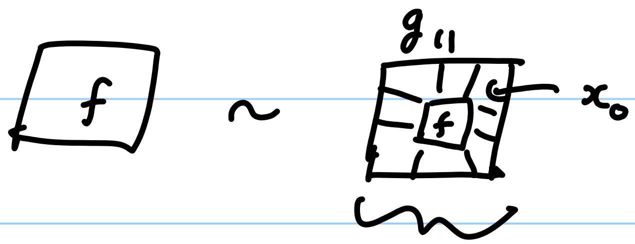


Theorem 0: $\pi_n(X, x_0)$ is a group.

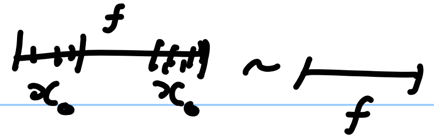
Pf: Just as with π_1.

Theorem 1: For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.

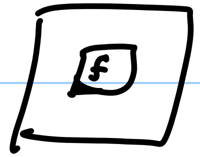
Pf: ·



(in π_1, $f \sim e * f * e$)



· Take a smaller square



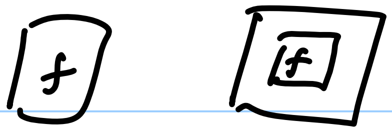
in I^n and define the map to be f rescaled on this square.

· Take the the constant map x_0 outside the smaller square.

Explicitly: Let $I = [-1, 1]$.

The map is $$g(\underset{\sim}{x}) = \begin{cases} f(4x), & |x_i| \leq 1/4 \ \forall i \\ x_0 & \text{otherwise.} \end{cases}$$

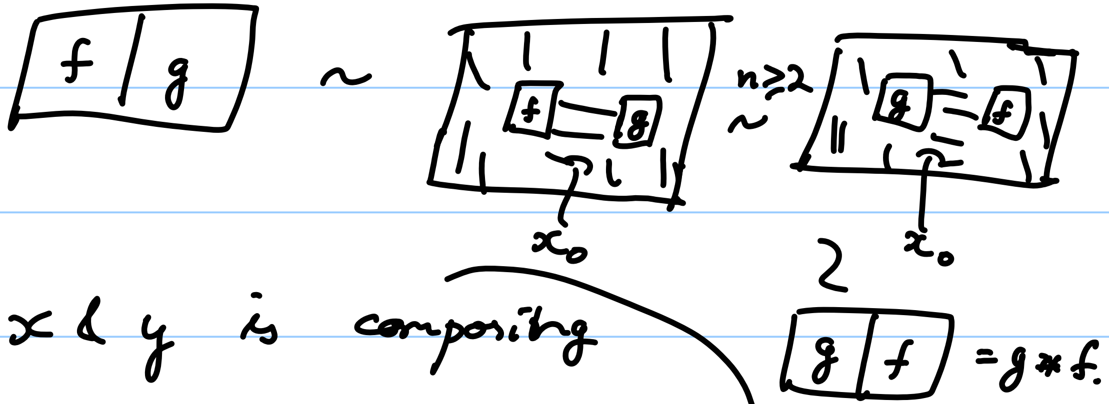
- A homotopy from g to f is



$$H(x,t)=\begin{cases} f\left(\frac{4}{1+3t}x\right), & \left|\frac{4}{1+3t}x_i\right| \leq 1 \ \forall i \\ x_0 & \text{otherwise} \end{cases}$$

$\frac{1}{4}+\frac{3t}{4}=\frac{1+3t}{4}$

- Hence $f * g =$



$= g * f$.

- Formally, ~~f~~flipping x & y is composing with a family of homeomorphisms $\varphi_t : I^n \to I^n$ exchanging f & g (and fixing ∂I^n) ($\varphi_0 = \mathrm{id}$, φ_1 exchanges small cubes)

∎

- Given $f : (X, x_0) \to (Y, y_0)$, we have induced homomorphisms $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$.

Theorem 2: Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Then $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$.

Pf: We use homotopy lifting.

Surjectivity:

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d] \\
(I^n, \partial I^n) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0)
\end{tikzcd}$$

by map lifting (using $\tilde{f}(\partial I^n) \subset p^{-1}(x_0)$ which is discrete, and ∂I^n is connected)

- $[f] = p_*([\tilde{f}])$.

Injectivity: Lift homotopies. □

Theorem: $\pi_n(T^2) = 0$ for $n \geq 2$.

Pf: $\pi_n(T^2) = \pi_n(\mathbb{R}^2)$

- $\pi_n(\mathbb{R}^2) = 0$ as $\mathbb{R}^2$ is contractible, using

Lemma: If $f, g : (X, x_0) \to (Y, y_0)$ are homotopic as maps between based spaces, then $f_* = g_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$

$\square$

Defn: A space X is aspherical if $\pi_n(X) = 0 \;\forall n \geq 2$.

$\Leftrightarrow$ $\pi_n(\tilde{X}) = 0 \;\forall n$, where $\tilde{X}$ is the universal cover.

Propn: $\pi_2(S^2) \neq 0$.

Pf: Consider the identity map

$$id: \overset{I^2/\partial I^2}{\overset{\shortparallel}{S^2}} \longrightarrow S^2$$

giving an element in $\pi_2(S^2)$

- We show id is not homotopic to the constant map

Namely, $\overset{= id}{id_*}: H_2(S^2) \longrightarrow H_2(S^2)$ is different from the map induced by a constant. □

Hurewicz homomorphism: There is a homomorphism

$$\varphi: \pi_n(X, x_0) \longrightarrow H_n(X)$$

Namely, a class $[f] \in \pi_n(X, x_0)$ gives a map

$$\bar{f} : S^n = I^n / \partial I^n \longrightarrow X.$$

- $H_n(S^n) = \mathbb{Z}$, with generator $[S^n]$
- Define $\varphi([f]) = \bar{f}_*(\underset{H_n(S^n)=\mathbb{Z}}{\underset{\uparrow}{[S^n]}}) \in H_n(X)$

- This is well-defined.

E.g. $H_2(T^2) = \mathbb{Z}$ (we will see this)

but $\pi_2(T^2) = 0$, so φ may not be surjective.

- In general, φ is not injective.

Theorem 3(Hurewicz theorem)

Suppose for $1 \leq k < n$, $\pi_k(X, x_0)$ is trivial

Then $\varphi : \pi_n(X, x_0) \to H_n(X)$ is an isomorphism.

E.g. If $\tilde{X}$ is the universal cover of X, then

$$\pi_2(X, x_0) = H_2(\tilde{X}).$$

Propn: $\pi_n(S^n) = \mathbb{Z}$, $n \geq 2$ & $\pi_k(S^n)$ is trivial for $k<n$.

Pf: $\pi_1(S^n) = 0$.

We show inductively that $\pi_k(S^n) = 0$

- Suppose $1 < k < n$,
  - Suppose $\pi_m(S^n)$ is trivial for $1 \leq m < k$
  - By Hurewicz theorem,

$$\pi_k(S^n) = H_k(S^n) = 0.$$

  - Thus, if $k < n$, $\pi_k(S^n)$ is trivial.
  - By Hurewicz theorem, $\pi_n(S^n) = H_n(S^n) = \mathbb{Z}$.

$\square$

- A similar argument shows:

<u>**Propn:**</u> If $\tilde{X}$ is the universal cover of X,

X is aspherical iff $\tilde{H}_*(\tilde{X}) \equiv 0$

Theorem: $\pi_3(S^2) = \mathbb{Z}$.

Some ideas:

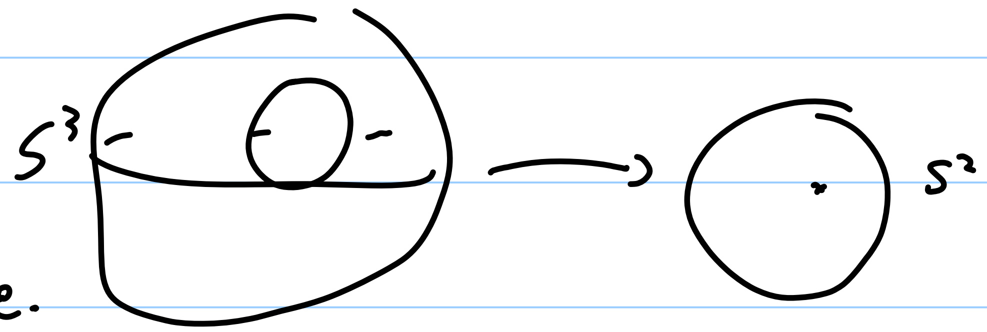
- The map $h: S^3 \to S^2$ is the *Hopf fibration.*
  - $S^3 \subseteq \mathbb{C}^2 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$
  - $S^2 = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$

The Hopf map is $h : (z_1, z_2) \mapsto z_1 / z_2 .= [z_1 : z_2]$

- If $h(z_1, z_2) = z$,

$h^{-1}(z) = \{(\alpha z_1, \alpha z_2]; \alpha \in \mathbb{C}, |\alpha| = 1\}$

which is a circle.



- In other words,

  S^1 acts on S^3 by $\alpha \cdot (z_1, z_2) = (\alpha z_1, \alpha z_2)$

  and the quotient is S^2.

  - This is written as $$\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}$$

- h is homotopically non-trivial because the inverse images $h^{-1}(p)$ & $h^{-1}(q)$ for $p \neq q$ form a 'Hopf link' in S^3

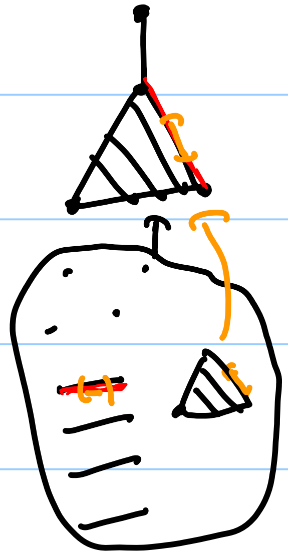
28/3/11

Lecture 20: CW-complexes and cellular homology

- CW-complexes are spaces inductively constructed by attaching 'k-cells': images of k-discs.

Defn: A CW-complex is a space X together with a collection of 'characteristic maps'

$$\{\varphi_\alpha : \overset{\overset{D_\alpha}{\shortparallel}}{D_\alpha^{k_\alpha}} \longrightarrow X\}_{\alpha\in A},\ D_\alpha^{k_\alpha} \text{ a } k_\alpha\text{-disc, s.t.}$$



(a) $X = \coprod_{\alpha\in A} \varphi_\alpha(\mathring{D}_\alpha)$ as a set; $\varphi_\alpha|_{\mathring{D}_\alpha}$ is 1-1.

(b) Let $X^{(n)} = \bigcup_{k_\alpha\leq n} \varphi_\alpha(D_\alpha^{k_\alpha})$ - the 'n-skeleton';

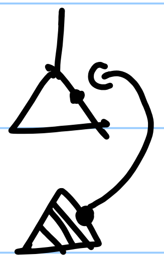
then if $k_\alpha = k$, $\varphi_\alpha(\partial D_\alpha) \subset X^{(k-1)}$.

(c) X has the weakest topology s.t. all φ_α are continuous.

CW-topology: $U \subset X$ is open $\Leftrightarrow$ $\varphi_\alpha^{-1}(U) \subset D_\alpha$ is open $\forall \alpha \in A$

Inductive construction of X: First 'view X, φ_α inductively

- $X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \dots$; $X = \bigcup X^{(n)}$
- In fact, '$X = \varinjlim X^{(n)}$'; i.e., $U \subset X$ is open iff $U \cap X^{(n)}$ is open $\forall n$.



- $X^{(0)}$ is a discrete set.
- Let $\theta_\alpha : \partial D_\alpha^{k_\alpha} = S_\alpha^{k_\alpha - 1} \longrightarrow X^{(k_\alpha - 1)}$ be maps.

  Then $X^{(n)} = X^{(n-1)} \cup \left(\bigcup_{k_\alpha = n} \varphi_\alpha(\mathring{D}_\alpha) \right)$ as a set
- As a space: $X^{(n)} = \left(X^{(n-1)} \cup \left(\coprod_{k_\alpha = n} D_\alpha \right) \right) \Big/ \sim$ 'formal disjoint union'

  with $\sim$ generated by $x \in \partial D_\alpha \sim \theta_\alpha(x) \in X^{(n-1)}$.

Inductive description of CW-complexes:

$A = \coprod_{n \geq 0} A_n$, A_n - index set for n-cells.

Given: · A collection of n-cells $\{D^n_\alpha\}_{\alpha \in A_n}$, formally we are given a set A_n & consider the space

$D^n \times A_n$, A_n with discrete topology.

$D^n_\alpha = D^n \times \{\alpha\}$

· $X^{(n-1)}$ = CW-complex consisting of cells of dimension $\leq n-1$.

· 'Attaching maps' $\theta_\alpha : \partial D^n_\alpha \longrightarrow X^{(n-1)}$

Define $X^{(n)} = \left(X^{(n-1)} \coprod \Big(\underset{\overset{\|}{D^n \times A_n}}{\coprod} D^n_\alpha \Big) \right) \Big/ \sim$; $x \in \partial D_\alpha \sim \theta(x) \in X^{(n-1)}$

- Inductively this gives $X^{(n)}$ as a topological space and characteristic maps for $\beta \in A_n$

$$\varphi_\beta : D_\beta^n \longrightarrow \left(X^{(n-1)} \cup \left(\coprod_{\alpha \in A_n} D_\alpha \right) \right) / \sim$$

by the composition of $id : D_\beta \to D_\beta$ with the quotient map.

Qn: What is $\sim$? (Equivalence relation generated by

$$x \in \partial D_\alpha \sim \Theta(x) \in X^{(n-1)}$$

Claim: $x \sim y$ iff one of these happen for (x,y) or (y,x)

$$\begin{cases} (i)\ x \in \mathring{D}_\alpha^n \text{ and } x = y. \\ (ii)\ x \in \partial D_\alpha \text{ and } y = \Theta_\alpha(x) \in X^{(n-1)} \\ (iii)\ x \in \partial D_\alpha,\ y \in \partial D_\beta \text{ and } \Theta_\alpha(x) = \Theta_\beta(y) \\ \qquad \text{(including } \alpha = \beta) \\ (iv)\ x, y \in X^{(n-1)}\ \&\ x = y \end{cases}$$

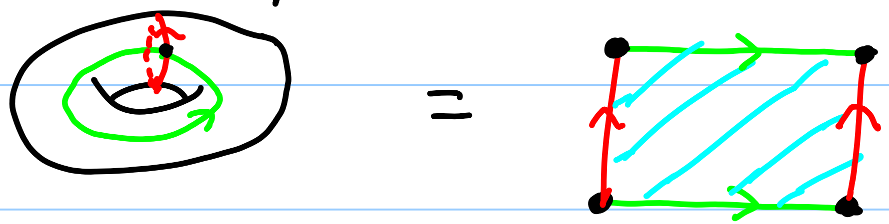
Proof of claim: Observe that the relation given above: (a) includes $x \sim \theta_\alpha(x)$ $\forall x \in D_\alpha$

(b) is reflexive, symmetric and transitive.

(c) Is the smallest relation satisfying (a) & (b)

Exercise: see this after an example

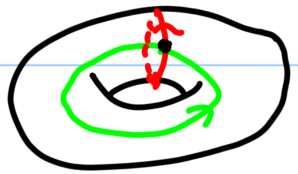
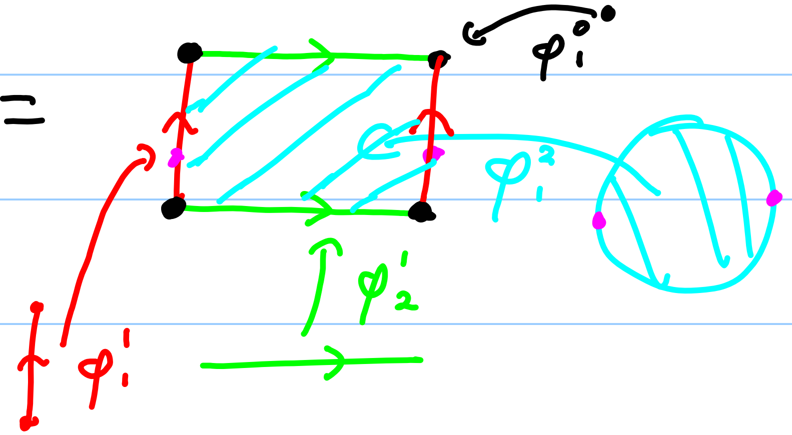
Example: CW complex for a torus:

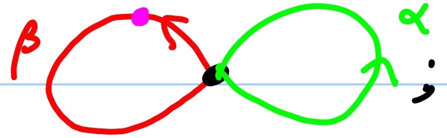
$X =$ 

- We have a CW-complex structure with

$$\begin{cases} \cdot\ 1 \quad 0\text{-cell} \quad \bullet \\ \cdot\ 2 \quad 1\text{-cells} \quad \to \quad \to \\ \cdot\ 1 \quad 2\text{-cell} \quad ▨ \end{cases}$$

Characteristic maps: The obvious maps.

$X =$  $=$ 

- $T^2 = \coprod \varphi_\alpha(\mathring{D}_\alpha)$ as a set
- $X^{(0)} = \bullet$; $X^{(1)} =$  ;

- 'Attaching maps' $\theta_1^1, \theta_1^2 : \underbrace{\partial D^1}_{\vdash\!\dashv} \longrightarrow X^{(0)}, \bullet\text{---}\bullet \mapsto \bullet$

  $X^{(1)} = \big(\overset{X^{(0)}}{\bullet} \sqcup$  $\sqcup$  $\big)/\sim$

- $X^{(2)} =$ torus ; $\theta_2^2 : S^1 \longrightarrow X^{(1)}$ is $\alpha * \beta * \bar{\alpha} * \bar{\beta}$.

30/3/2011

Lecture 21: CW-complexes contd.

Homotopy of attaching maps:

Suppose X is a topological space,

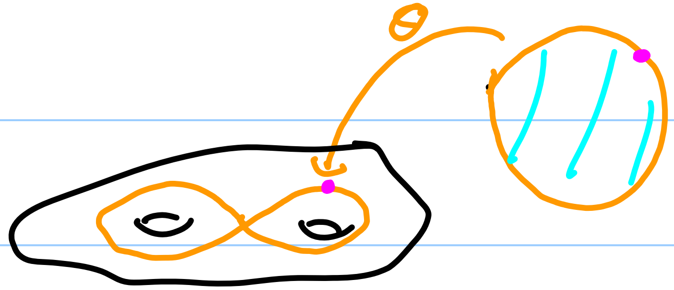
$\theta : \partial D^n \longrightarrow X$ is a map

· The space Y obtained from X by attaching D^n along θ is:

$$Y = (X \amalg D^n)/\sim$$

where $\sim$ is generated by

$$p \in \partial D^n \sim \theta(p) \in X$$



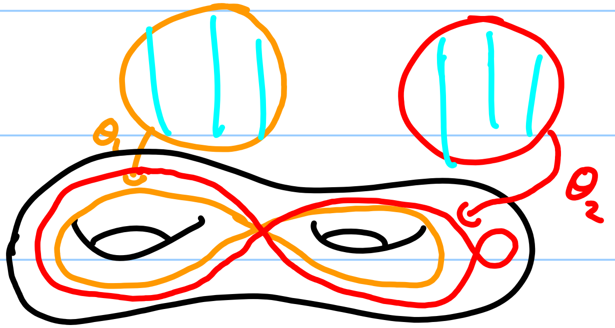
Theorem (Whitehead):

Suppose $\theta_1, \theta_2 : \partial D^n \to X$ are homotopic maps and Y_i is obtained from X by attaching D^n using θ_i, then Y_1 is homotopy equivalent to Y_2.

Pf: We shall construct maps

$$\varphi_i : Y_i \longrightarrow Y_{3-i}, \quad i=1,2$$

and homotopies $\varphi_i \circ \varphi_{3-i} \sim id$.



Construction of $\varphi_1 : Y_1 \longrightarrow Y_2$: $Y_i = (X \amalg D^n)/_{p \sim \theta_i(p)}$

- We define $\tilde{\varphi}_1 : X \amalg D^n \to Y_2$ which induces $\varphi : Y_1 \to Y_2$

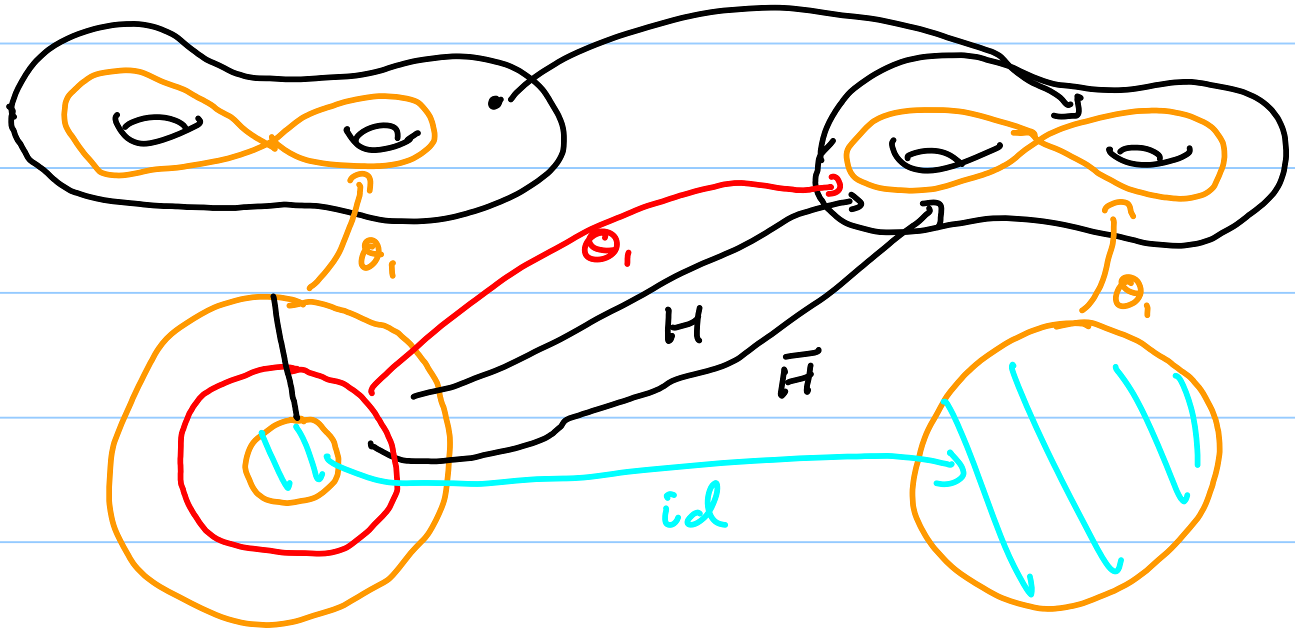
- $\tilde{\varphi}$ on X is the composition of identity with the quotient map $X \amalg D^n \longrightarrow Y_2$.
- Let H be the homotopy from θ_1 to θ_2. We use this to define $\tilde{\varphi}$ on $D^n = A \cup \hat{D}^n$

- $D^n = \hat{D}^n \cup (\partial D^n \times I)$, $H : \partial D^n \times I \to X$
- We define $\tilde{\varphi}_1$ on $\hat{D}_n$ to be the 'scaling' homeomorphism $\hat{D}_n \to D_n$ composed with the quotient.
- We define $\tilde{\varphi}_1$ on $\partial D^n \times I$ to be H.
- <u>Convention</u>: $H|_{\partial D^n \times \{0\}} = \theta_1$, $H|_{\partial D^n \times \{1\}} = \theta_2$

  and we identify $\begin{cases} \cdot\ \partial D^n \times \{0\} \text{ with } \partial D^n. \\ \cdot\ \partial D^n \times \{1\} \text{ with } \partial \hat{D}^n. \end{cases}$

- Then $\tilde{\varphi}_1$ is continuous and induces a continuous map $\varphi_1 : Y_1 \to Y_2$.
- $\varphi_2 : Y_2 \to Y_1$ is defined similarly (using $\bar{H}$)

- $\varphi_2 \circ \varphi_1 : Y_1 \to Y_1$
- $\underline{\varphi_2 \circ \varphi_1 \sim id}$: $\varphi_2 \circ \varphi_1$ on X is the identity



Now, use the homotopy '$H * \bar{H} \sim id$.'

Recall: A space X is aspherical if $\pi_i(X)=0 \ \forall i \geq 2$

$\iff$ The universal cover $\tilde{X}$ satisfies $\pi_i(\tilde{X})=0 \ \forall i \geq 1$.

Theorem (Whitehead): If Y is a path-connected CW-complex such that $\pi_i(Y)$ is trivial $\forall i \geq 1$, then Y is contractible.

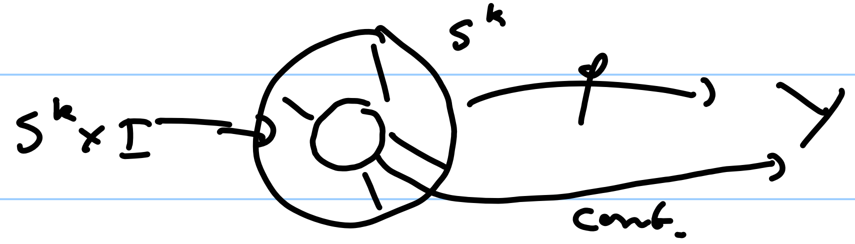
Rk: The converse is clearly true.

Rk: By Hurewicz theorem, the hypothesis is equivalent to $\pi_1(Y)=1$ & $H_i(Y)=0 \ \forall i$.

Proof: We use the following lemma.

Lemma: If $\pi_k(Y)$ is trivial, then any map $\varphi: S^k \to Y$ extends to a map $\varphi: D^{k+1} \to Y$.

Pf: Follows from the definition of π_k and the observation $S^k \times [0,1] / S^k \times \{0\} = D^{k+1}$.

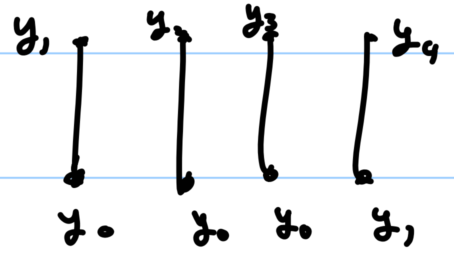


- Namely φ gives an element of $\pi_k(Y) = 0$, so φ is homotopic to a constant map. The homotopy $H: S^k \times I \to Y$ induces $\varphi: D^{k+1} \to Y$ □

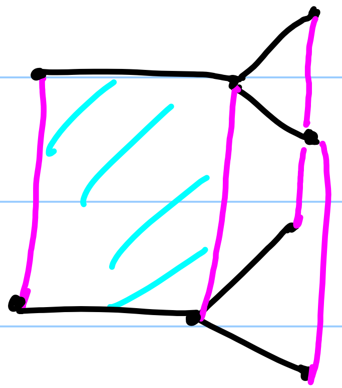
<u>Pf of theorem:</u> Y is a CW-complex, $Y = \bigcup Y^{(n)}$.

- We construct $H : Y \times [0,1] \longrightarrow Y$, a homotopy from the constant map $Y \to \{y_0\}$ to the identity.
- We construct $H : Y^{(n)} \times [0,1] \to Y$ inductively on n.
- $\underline{Y^{(0)} \times [0,1]} = \underline{\coprod} \{y_i\} \times [0,1]$.

For each $\{y_i\}$, define $H : \{y_i\} \times [0,1] \to Y$ to be a path from y_0 to y_i



- We extend from $Y^{(0)} \times I$ to $Y^{(1)} \times I$



- $H : Y^{(1)} \times I \to Y$ is specified on $Z^{(0)} = \left\{ \begin{array}{ll} \cdot\ Y^{(0)} \times I & (\text{induction}) \\ \cdot\ Y^{(1)} \times \{0\} & (\text{constant}) \\ \cdot\ Y^{(1)} \times \{1\} & (\text{identity}) \end{array} \right\}$
- $Y^{(1)} \times I$ is obtained from $Z^{(0)}$ by attaching discs $D_k^2 = I \times I$, and H has been defined on ∂D_k^2. Further interiors of D_k^2 are disjoint.
- By lemma, H extends to each D_k^2, hence $Y^{(1)} \times I$.

- Inductive extension from $Y^{(n)} \times I$ to $Y^{(n+1)} \times I$ is similar.
- We thus get $H : Y \times I \to Y$ using $Y = \varinjlim Y^{(n)}$. $\square$

Theorem (Whitehead$\overset{?}{}$(Hopf?)): If X & Y are aspherical CW-complexes with $\pi_1(X) \cong \pi_1(Y)$, then $X \overset{h.e.}{\sim} Y$.

Consequence: $H_*(X)$ is determined by $\pi_1(X)$.

Hopf thus defined 'group (co)homology' of G the (co)homology of an aspherical CW-complex X s.t. $\pi_1(X) = G$. $\square$

Lecture 22:

Milnor's Universal Join Construction:

Theorem: Let G be a countable group (with discrete topology). Then there is an aspherical CW-complex X s.t. $\pi_1(X) = G$.

Rk: By Whitehead's theorem, X is unique up to homotopy equivalence.

Hopf: $H_*(G) := H_*(X)$.

- We actually construct $\tilde{X}$ contractible CW-complex s.t. $G \circlearrowright \tilde{X}$ as deck transformations. $X = \tilde{X}/G$.

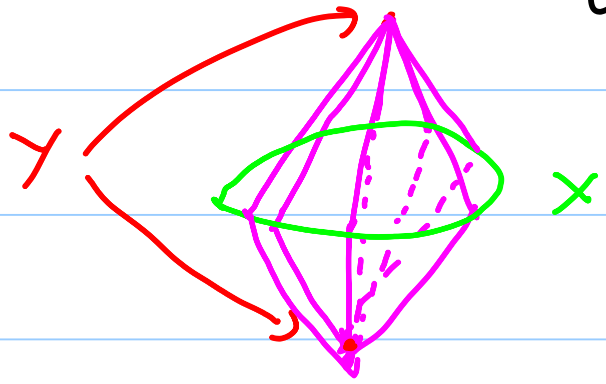
Join: Let X & Y be topological spaces

Then the join $X * Y$ is

$$X \times Y \times [0,1] / \sim$$

with $(x, y, t) \sim (x', y', t')$ iff one of the following holds:

$$\begin{cases} \cdot\ x = x',\ y = y' \ \& \ t = t' \\ \cdot\ t = t' = 0 \text{ and } x = x' \\ \cdot\ t = t' = 1 \text{ and } y = y' \end{cases}$$



Informally: Join each point of X to each point of Y by an interval $[0,1]$

- We identify $x_0 \in X$ with $[(x_0, y, 0)]$ in $X * Y$, independent of y.

- The path (x, y, t), $t \in [0,1]$ joins $(x, y, 0) = x \in X$ to $(x, y, 1) = y \in Y$ under the identifications given by: $x_0 \mapsto [(x_0, y, 0)] \in X * Y$ (independent of y)
$y_0 \mapsto [(x, y_0, 1)] \in X * Y$ (independent of x).

<u>Rk:</u> $S^n * S^m = S^{n+m+1}$.

<u>Propn:</u> Suppose $X, Y \neq \emptyset$. Then any map $f : S^n \longrightarrow X$ extends to $f : D^{n+1} \longrightarrow X * Y$.

<u>Pf:</u> Pick $y_0 \in Y$ and define $f : \overset{= S^n \times [0,1]/(S^n \times \{0\})}{D^{n+1}} \longrightarrow X * Y$ as induced by $f(p,t) = \overset{\in X*Y}{(f(p), y_0, 1-t)}$ (const. y_0 on $S^n \times \{0\}$) $\square$

Cor: $\pi_n(X) \longrightarrow \pi_n(X * Y)$ is the zero map.

Rk: $X * S^0 = \Sigma X$ is called the suspension of X and we have $\pi_{n+1}(\Sigma X) = \pi_n(X)$ (with appropriate hypothesis)

- In particular, if $Y \neq \emptyset$, $X * Y$ is connected.

Milnor's construction:

- G acts on itself freely by left multiplication.
- $G \curvearrowright X$ and $G \curvearrowright Y$ freely $\Rightarrow$ $G \curvearrowright X * Y$ freely.

- Let $X_n = \underbrace{G * G * \dots * G}_{n \text{ copies}}$, $X_1 \subset X_2 \subset X_3 \subset \dots$
- $X = \varinjlim X_n$, i.e.,

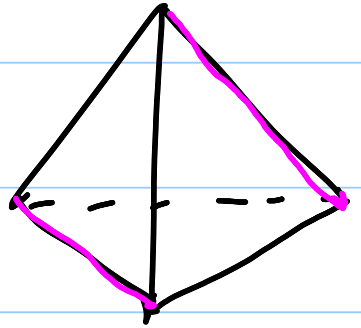
$X = \bigcup X_i$ as a set

and $U \subset X$ is open iff $X_n \cap U$ is open in X_n $\forall n$.

Theorem (Milnor):

X is contractible.

Pf: $\cdot$ X has a CW-structure coming from its description as a join($D^k * D^n = D^{k+m+1}$, namely a simplex is the join of an opposite pair of spaces.)



- We construct $H : X^{(n)} \times [0,1] \longrightarrow X$ contraction inductively on n.
- For the inductive step, we need to extend maps from $S^n \longrightarrow X$ to $D^{n+1} \longrightarrow X$.
  - Each such relevant map maps into $X_N \subset X$ for some N (can show this by induction)
    - By the above, this extends to a map

$$D^{n+1} \longrightarrow X_{N+1} = \underset{\underset{\underbrace{G * \cdots * G}_{N \text{ times}}}{\parallel}}{X_N} * G. \qquad \square$$

- X/G gives the required CW-complex (called $K(G,1)$ or Eile-'

- The CW-complex is called a $K(G,1)$ or an Eilenberg-Maclane space.

<u>Action of G on X:</u> If $G \curvearrowright Z$ & $G \curvearrowright Y$ then

$$G \curvearrowright Z * Y \quad \text{by} \quad g\cdot(z, y, t) = (gz, gy, t).$$

- Each point $x \in X$ is in X_n for some n, and we get a well-defined action on X.

<u>On proper discontinuity:</u>

Suppose $G \curvearrowright X$ is not properly discontinuous

- Then $\exists K$ compact and $g_1, g_2, \dots$ all distinct such that

$$g_i K \cap K \neq \emptyset.$$

- This means $\exists\, x_i \in K$ s.t. $g_i x_i \in K$.
- On passing to a subsequence, we can assume that $\begin{cases} \cdot\ x_i \longrightarrow x \\ \cdot\ g_i x_i \longrightarrow x' \quad (\text{as } g_i x_i \in K) \end{cases}$

Thus, $\exists\, g_i$ distinct and points $x_i \in X$ s.t.

$$x_i \to x \quad \text{and} \quad g_i x_i \longrightarrow x'.$$

Conversely, if $x_i \to x$, $g_i x_i \to x'$,

let $K = \left(\bigcup_i \{x_i\} \right) \cup \left(\{x, x'\} \right) \cup \left(\bigcup_i \{g_i x_i\} \right)$

- This is compact.
- $g_i K \cap K \neq \emptyset \quad \forall i$

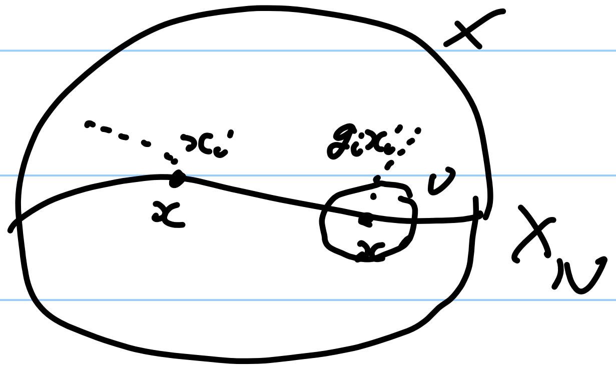
Hence the action is not properly discontinuous.

<u>Now,</u>

Suppose $X = \underrightarrow{\lim}\, X_n$, the action of G on each X_n is properly discontinuous.

- If $G \curvearrowright X$ is <u>not</u> properly discontinuous, $\exists x, x', g_i, x_i$ as above.

$\exists N$ s.t. $x, x' \in X_N$.



~~· Assume appropriate 'uniform continuity'.~~

~~· Consider the sequence $g_i x \in X_N$. By proper discontinuity, we cannot have $g_i x \to x'$.~~

~~· 'We conclude that $g_i x_i \not\to x'$.'~~

In $X = G * G * \cdots$

$$x = (\gamma_1, t_1, \ldots, \gamma_n, t_n, e, 0, e, 0 \ldots)$$

$$x' = (\gamma_1', t_1, \ldots, \gamma_n', t_n', e, 0, e, 0, \ldots)$$

$$x_i = (\gamma_{1i}, t_1, \gamma_{2i}, t_2 \ldots \quad)$$

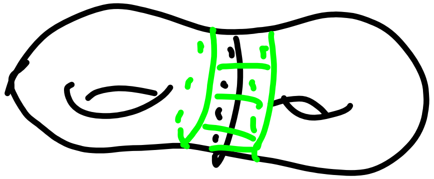
$$g_i x_i = (g_i \gamma_{1i}, t_{1i}, \ldots \quad) \to x'$$

So $\left\{ \begin{array}{l} g_i \gamma_{1i} \to \gamma_1' \\ \gamma_{1i} \to \gamma_1 \end{array} \right\}$ or $t_1 = 1$

$\Rightarrow g_i$ are a finite set (by proper discontinuity)

- We proceed inductively.

$\square$



Lecture 23: Cellular homology

Relative Homology: $H_n(X, A)$

Theorem: X is a space, $A \subset X$ closed subset s.t. $\exists\, U \subset X$ open, $A \subset U$ s.t. U deformation retracts onto A. Then $H_*(X, A) \cong \tilde{H}_*(X/A)$, where X/A is the quotient obtained by identifying A to a point.

Proof: (1) $H_*(X, A) \cong H_*(X, U)$ with the isomorphism induced by inclusion

Pf: $H_*(A) \xrightarrow{\cong} H_*(U)$ as U defm. retracts to A.

We have,

$$\begin{tikzcd}
\arrow[r] & H_n(A) \arrow[r] \arrow[d, "\cong"] & H_n(X) \arrow[r] \arrow[d, "\cong"] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(X) \arrow[r] \arrow[d, "\cong"] & {} \\
\arrow[r] & H_n(U) \arrow[r] & H_n(X) \arrow[r] & H_n(X,U) \arrow[r] & H_{n-1}(U) \arrow[r] & H_{n-1}(X) \arrow[r] & {}
\end{tikzcd}$$

· By 5 lemma, $H_n(X,A) \xrightarrow{\cong} H_n(X,U)$.

(2) U/A deformation retracts to a point $\{A\}$ (Exercise)

(3) Hence, $H_*(X/A, U/A) \cong H_*(X/A, \{A\})$

(4) $H_*(X/A, \{A\}) = \tilde{H}_*(X/A)$ by the long exact sequence

$$\underset{\substack{\shortparallel \\ 0}}{\tilde{H}_n(\{A\})} \longrightarrow \tilde{H}_n(X/A) \longrightarrow \tilde{H}_n(X/A, \{A\}) \longrightarrow \overset{\substack{0 \\ \shortparallel}}{\tilde{H}_{n-1}(\{A\})}$$

Summary: $H_*(X,A) \cong H_*(X,U)$; $H_*(X/A, U/A) \cong \tilde{H}_*(X/A)$

(5) $H_*(X,U) \cong H_*(X \setminus A, U \setminus A) \cong H_*(X/A \setminus \{A\}, U/A \setminus \{A\}) \cong H_*(X/A, U/A)$ $\square$

Cellular homology: Let X be a CW-complex.

We define $C_n^{CW}(X) = H_n(X^{(n)}, X^{(n-1)})$

(we show) $\quad = $ Free abelian group on n-cells

$\partial_n : C_n^{CW}(X) \longrightarrow C_{n-1}^{CW}(X)$

$$\begin{tikzcd}
{} \arrow[r] & H_n(X^{(n)}, X^{(n-1)}) \arrow[r] & H_{n-1}(X^{(n-1)}) \arrow[r] \arrow[d, "\cong"] & \cdots & \\
 & \cdots \arrow[r] & H_{n-1}(X^{(n-1)}) \arrow[r] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[r] & \cdots
\end{tikzcd}$$

- ∂_n is the composition $H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)})$.

Propn: $\partial_{n-1} \circ \partial_n = 0$

Pf: We write $\partial_{n-1} \circ \partial_n$ in terms of 3 exact sequences.

$$\begin{tikzcd}
{} \arrow[r] & H_n(X^{(n)}, X^{(n-1)}) \arrow[r] & H_{n-1}(X^{(n-1)}) \arrow[r] \arrow[d, "\cong"] & {} & & \\
& & H_{n-1}(X^{(n-1)}) \arrow[r] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[r] & H_{n-2}(X^{(n-2)}) \arrow[d] & \\
& & & & H_{n-2}(X^{(n-2)}) \arrow[r] & H_{n-2}(X^{(n-2)}, X^{(n-3)})
\end{tikzcd}$$

- $\partial_n \circ \partial_{n-1}$ has two succesive terms of an exact sequence, so equals 0.

Definition: The cellular homology of X is the homology of $(C_*^{CW}(X), \partial_*^{CW})$

Theorem: Cellular homology = singular homology
$\wr$
simplicial homology

Pf: See Hatcher

Concrete view of C_*^{cw}:

(1) $H_k(X^{(n)}, X^{(n-1)}) \cong \tilde{H}_k(X^{(n)}/X^{(n-1)})$

(2) $X^{(n)}/X^{(n-1)} = \bigwedge S^n$, with a copy of S^n for each n-cell.

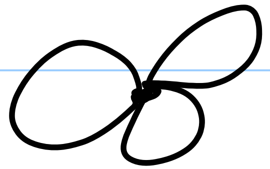
Pf: Let $\varphi_\alpha : D^n \longrightarrow X^{(n)}$ be a characteristic map.

This is injective on $\mathring{D}^n$.

We have an induced quotient

$$\bar{\varphi}_\alpha : \underset{\underset{S^n}{\shortparallel}}{D^n/\partial D^n} \longrightarrow X^{(n)}/X^{(n-1)}, \text{ mapping } \partial D^n \text{ to}$$

the point on all spheres S^n



- Together the maps φ_α give a homeomorphism

$$\bigwedge S^n \longrightarrow X^{(n)}/X^{(n-1)}.$$

$\square$

(3) $\tilde{H}_k(\bigwedge_{i=1}^{m} S^n) = \begin{cases} \mathbb{Z}^m & \text{if } k=n \\ 0 & \text{otherwise.} \end{cases}$

<u>Pf:</u> Inductively use Mayer-Vietoris & $\tilde{H}_*(pt)=0$ to conclude $\tilde{H}_k(\bigwedge_{i=1}^{m} S^n) \cong \bigoplus \tilde{H}_k(S^n)$.

Thus,

$C_n^{CW}(X)$ is the free abelian group generated by n-cells of X.

The boundary map:

Let $\varphi_\alpha : D^n \longrightarrow X$ be the characteristic map of an n-cell e_α ; $\psi_\beta : D^{n-1} \longrightarrow X$ be the char. map of an $(n-1)$-cell e_β.

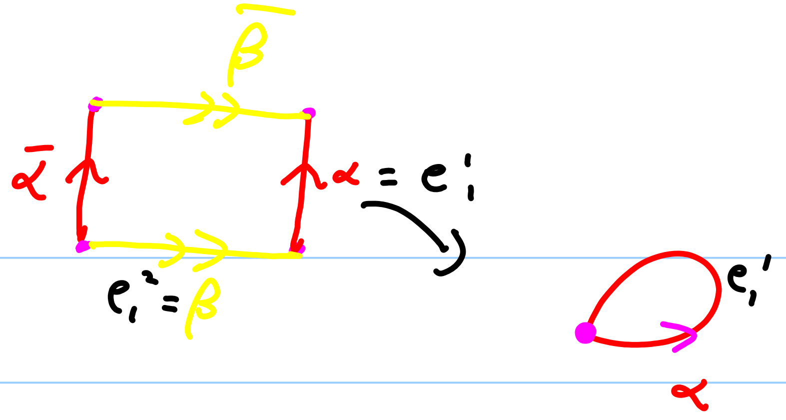
We compute: Coefficient of e_β in $\partial_n e_\alpha$.

- $X^{(n-1)} / (X^{(n-1)} \setminus \psi_\beta(\mathring{e}_\beta)) = S^{n-1}$

- Now $\varphi_\alpha|_{\partial D^n} : S^{n-1} \longrightarrow X^{(n-1)}$ so we get

$$S^{n-1} \xrightarrow[\varphi_\alpha|_{\partial D^n}]{} X^{(n-1)} \longrightarrow X^{(n-1)} / (X^{(n-1)} \setminus \psi_\beta(\mathring{e}_\beta)) = S^{n-1}$$

The coefficient is the degree of this map.



E.g. $X = T^2$:

$$\begin{tikzcd}
C_2 = \mathbb{Z} \arrow[d, "0"] & e^2 \\
C_1 = \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "0"] & (e_1^1, e_1^2) \\
C_0 = \mathbb{Z} & e^0
\end{tikzcd}$$

E.g. Coefficient of e_1^1 in ∂e_2 is the degree of a map $\underset{\partial D^2}{\overset{\shortparallel}{\varphi : S^1}} \longrightarrow S^1 = X^{(1)} / X^{(1)} \setminus \mathring{e}_1^1$; $\varphi = e * \alpha * e * \bar{\alpha} \sim e$

$\Rightarrow \deg(\varphi) = 0.$

$$H_k(T^2) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^2, & k = 1 \\ 0, & k \geq 3 \end{cases}$$