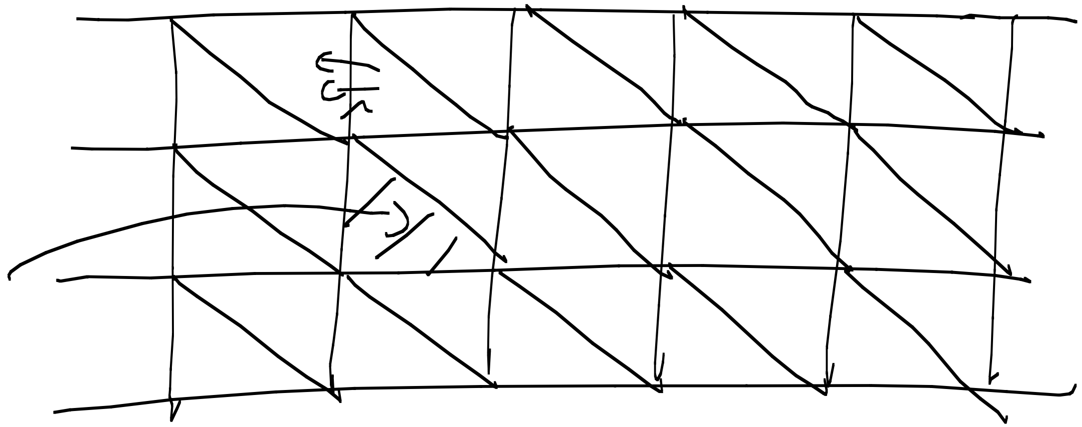
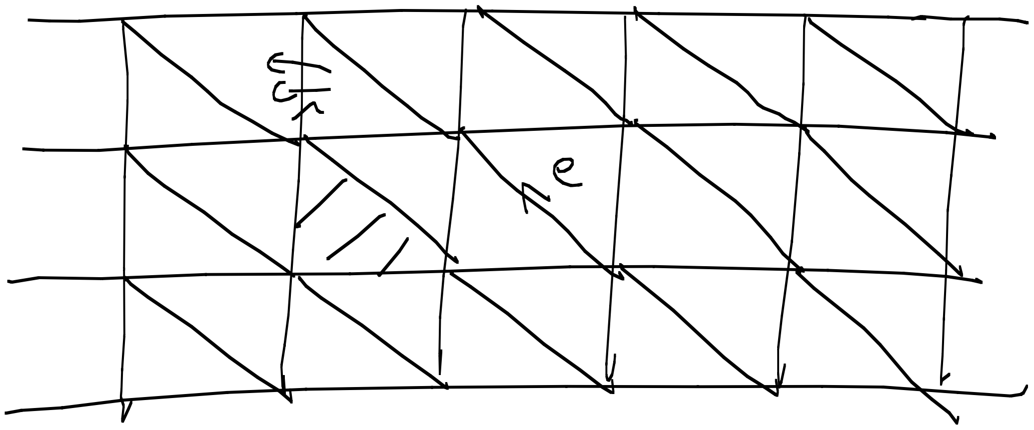
Algebraic topology

Chains, boundaries etc. 4/1/10

- Consider water flowing on the plane; subdivided into triangles.



- For each edge, we consider how much water flows through it.
- We orient the edges
- Consider a triangle. fHow be w much water flows into it?



* Given oriented edges, the flow can be regarded as a real number for each edge.

* We can also regard it as a linear combination of edges

$$C^1 = \{\varphi : e \longmapsto \varphi(e) \in \mathbb{R}\}$$

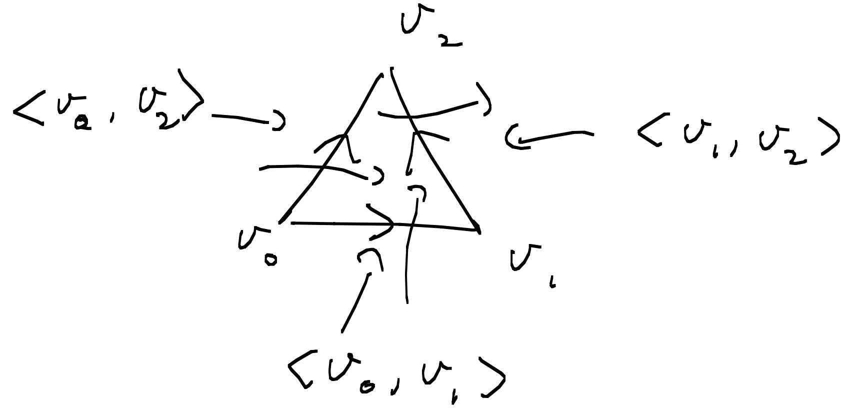
$$\wr\|$$

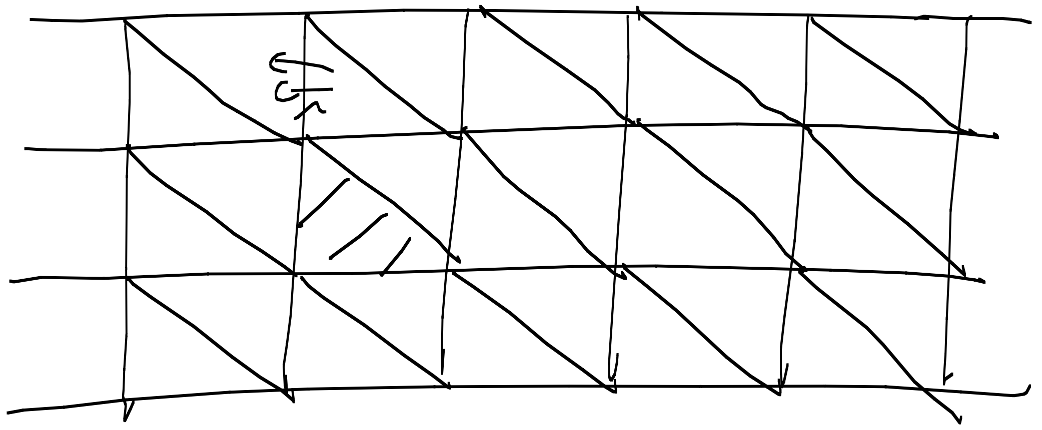
or $C_1 = \{\sum \varphi(e)\cdot e : \varphi(e) \in \mathbb{R}\}$

if the complex is finite

Water flowing ~~into~~ out of a triangle

$T = \langle v_0, v_1, v_2 \rangle$





Flow given by
$\varphi : e \mapsto \varphi(e)$
or $\xi = \sum \varphi(e) \cdot e$.

The flow out of the triangle is:

$$\delta\varphi(\langle v_0, v_1, v_2 \rangle) = \varphi(\langle v_1, v_2 \rangle) + \varphi(\langle v_0, v_1 \rangle) - \varphi(\langle v_0, v_2 \rangle)$$

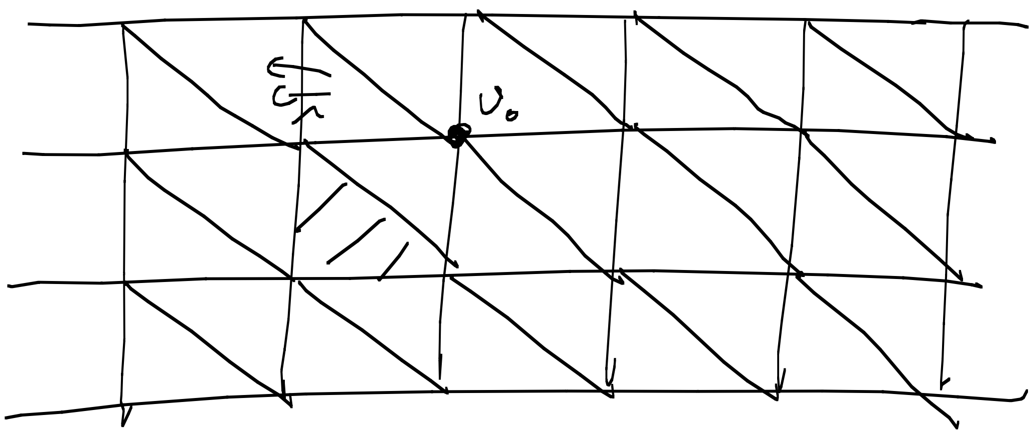
* $\delta\varphi$ is a function from triangles $\longrightarrow \mathbb{R}$.

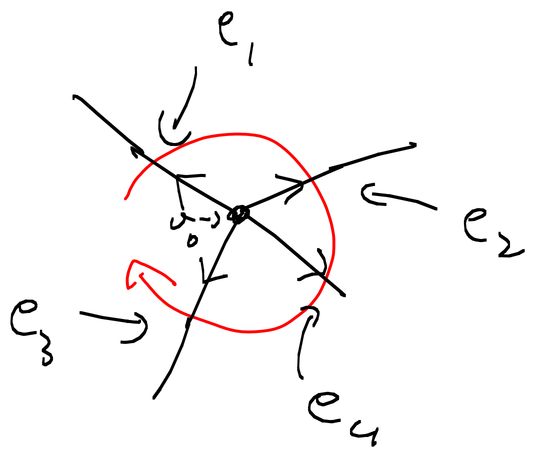
$$C^2 = \{ \varphi : \text{triangles} \longrightarrow \mathbb{R} \}$$
$$\cong$$
$$C_2 = \{ \sum \varphi(\tau) \cdot \tau : \tau \text{ triangle} \}$$

Thus, we have

$$C^2 \xleftarrow{\delta} C^1$$

$$\boxed{C^i \cong C_i}$$





- Let v_0 be a vertex, assume edges e_1, e_2, e_3, e_4 are oriented outwards

Let $\partial\varphi(v_0) = -[\varphi(e_1) + \varphi(e_2) + \varphi(e_3) + \varphi(e_4)]$

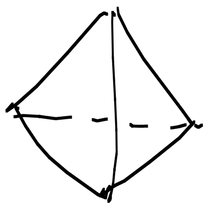
- $\partial\varphi(v_0)$ is the vorticity around v_0.
- $C_1 \xrightarrow{\partial} C_0$

Summary,

$$\begin{tikzcd}
C^2 & C^1 \arrow[l, "\delta"'] & C^0 \arrow[l, "\delta"'] \\
C_2 \arrow[u, phantom, sloped, "\cong"] \arrow[r, "\partial"] & C_1 \arrow[u, phantom, sloped, "\cong"] \arrow[r, "\partial"] & C_0 \arrow[u, phantom, sloped, "\cong"]
\end{tikzcd}$$

* Algebraic topology is based on chains : C_k, co-chains C^k and the boundary ∂ and coboundary δ.

<u>Three-dimensions:</u> Break space into tetrahedra.



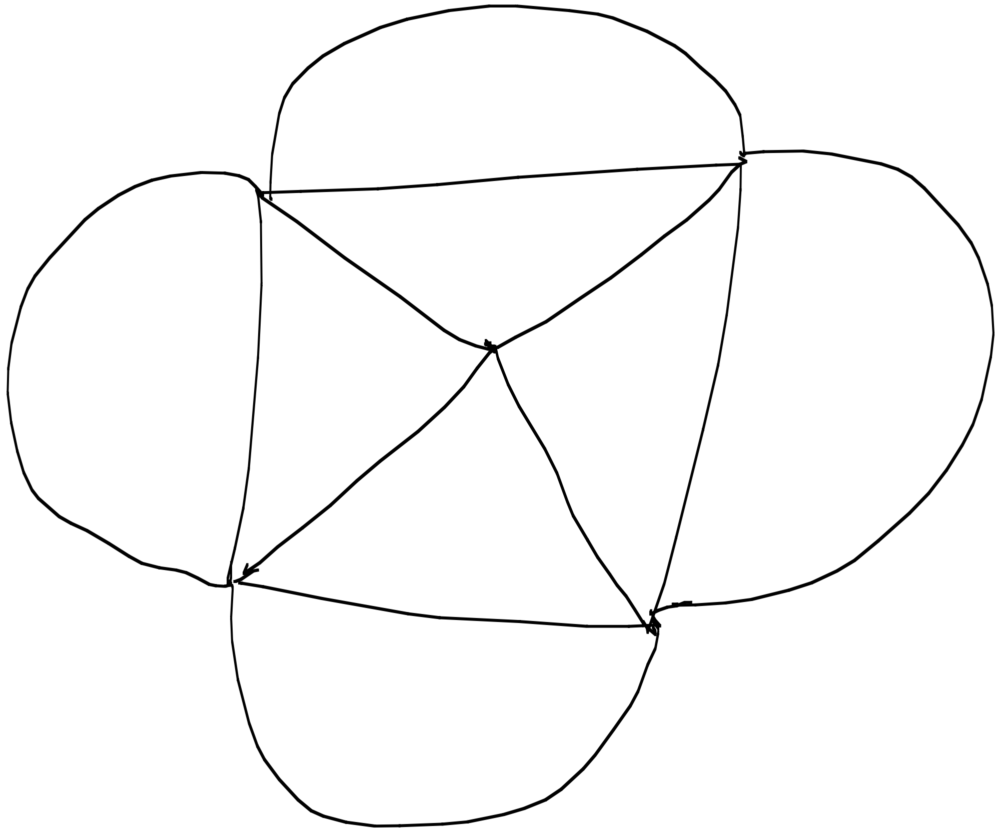
$$\boxed{\partial\circ\partial = 0}$$

* Flow determined by an element of $C_2 \cong C^2$

$$\begin{tikzcd}
C_3 \arrow[r, "\partial"] & C_2 \arrow[r, "\partial"] & C_1 \arrow[r, "\partial"] & C_0 \\
C_3 & C^2 \arrow[l, "\delta"'] \arrow[u, phantom, sloped, "\cong"] & C^1 \arrow[l] \arrow[u, phantom, sloped, "\cong"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\xi \arrow[r, mapsto] & \partial\xi \arrow[r, mapsto] & \partial\circ\partial\xi \\
C_2 \arrow[u, phantom, sloped, "\in"] & C_1 \arrow[u, phantom, sloped, "\in"] & C_0 \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

A puzzle

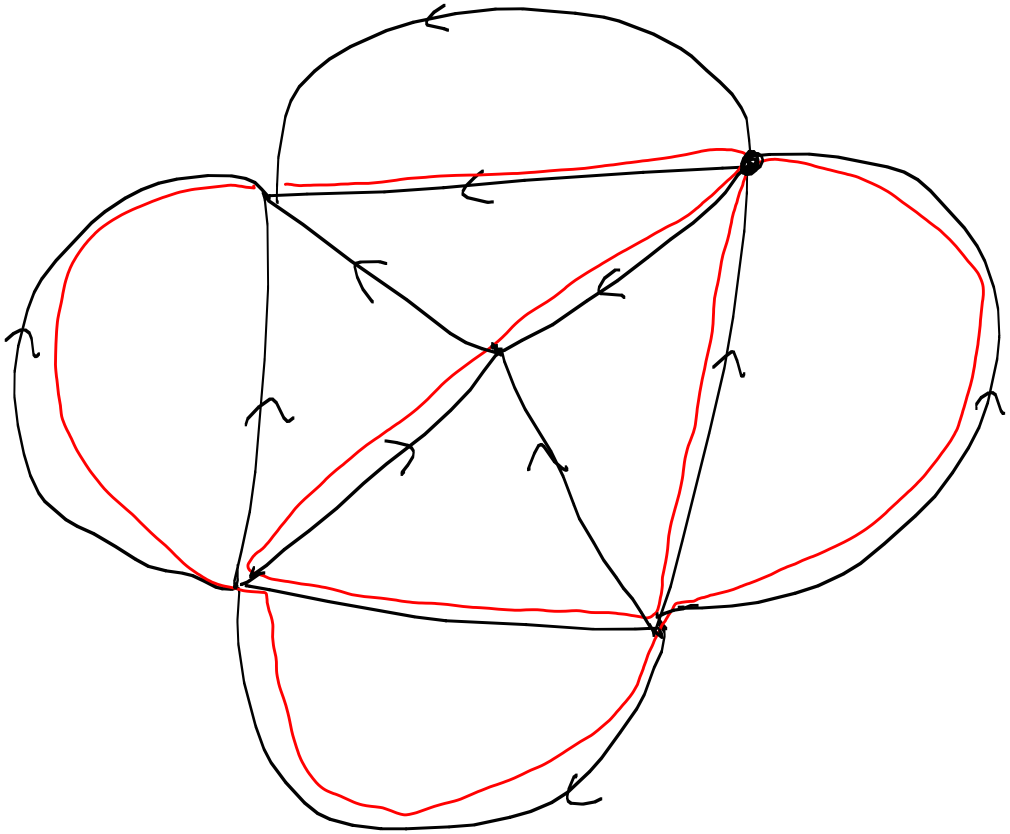


- * Draw the graph without repeating edges and without lifting the pen.

Ignore Order

- * Solution: Throw away information – Abelianise.

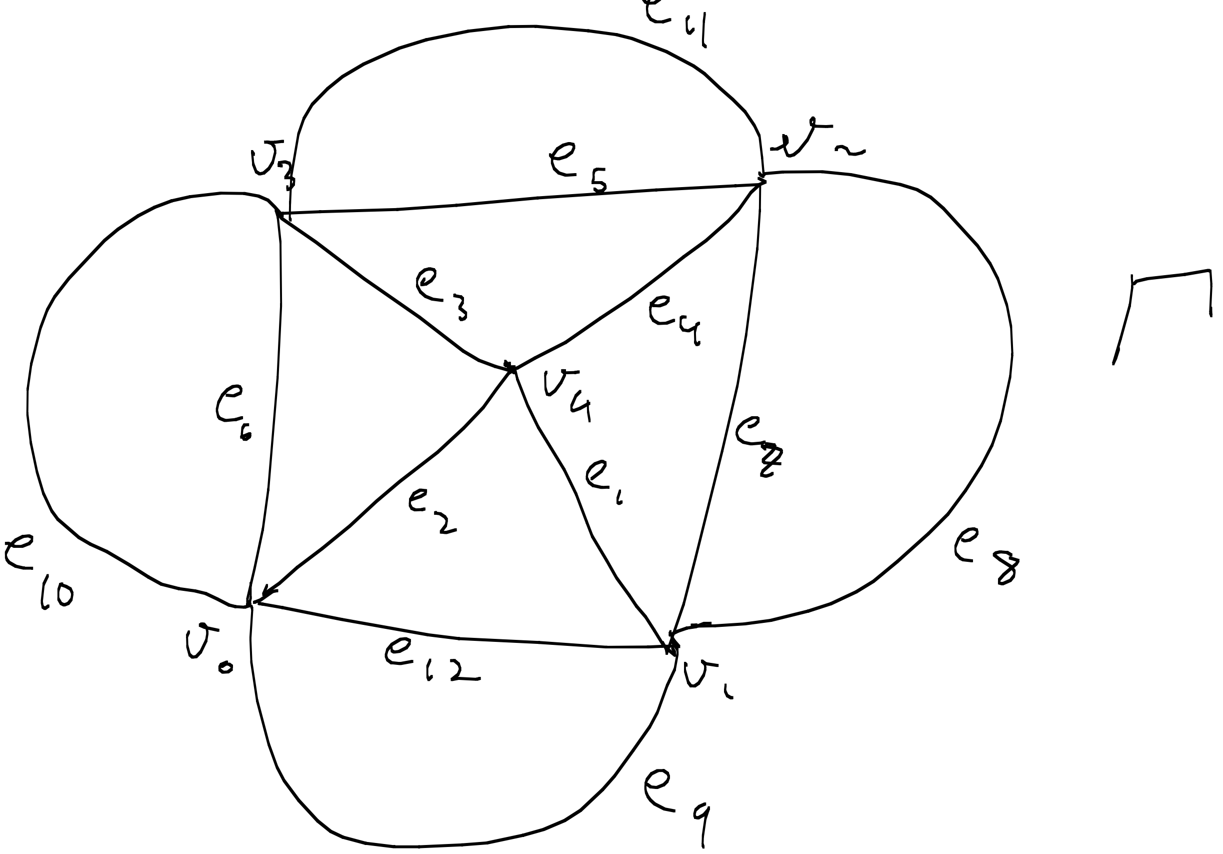
Consider a 'tour': We draw some edges without violating the conditions.



* Record the number of times each edge was crossed (with sign) ⟶ element of $C_1 \ni \xi$

* Observe: Legal path $\left\{ \begin{array}{l} \bullet\ \partial\xi = 0 \text{ or } \partial\xi = v_j - v_i \\ \bullet\ \text{Each coefficient is } \pm 1 \end{array} \right.$

Reduce modulo 2:



Contradiction.

$$\underset{\overset{\uparrow}{\xi}}{C_1}(\Gamma, \mathbb{Z}/_2) \longrightarrow C_0(\Gamma, \mathbb{Z}/_2)$$

- $\partial \xi = 0$ or $v_2 + v_1$ (2 terms)
- $\xi = e_1 + e_2 + \dots + e_{12} \Longrightarrow \partial \xi = v_0 + v_1 + v_2 + v_3$

Homology versus Homotopy groups.

Question: How do we generalise $\pi_1(\ )$?

E.g. To show $\mathbb{R}^3 \neq \mathbb{R}^4$

$$\begin{tikzcd} \pi_1 \sim \text{Maps from } S^1 \text{ (based maps)} \arrow[dr] \arrow[dd] & \\ & \pi_2 = \text{Maps from } S^2 \text{ (based maps)} \\ H_2 = \text{Maps from Surfaces} - \underline{\text{Easier}} & \end{tikzcd}$$

Reason: We have combination theorems

* Remark: $H_1 =$ Abelianisation of π_1
$\sim$ Maps from $\sqcup S^1$

Passage from π_1 to H_1

H_1 : Maps from $S^1 \amalg S^1 \amalg \ldots \amalg S^1$

- A loop without basepoint gives a *conjugacy class* in π_1

- We *cannot* multiply conjugacy classes,

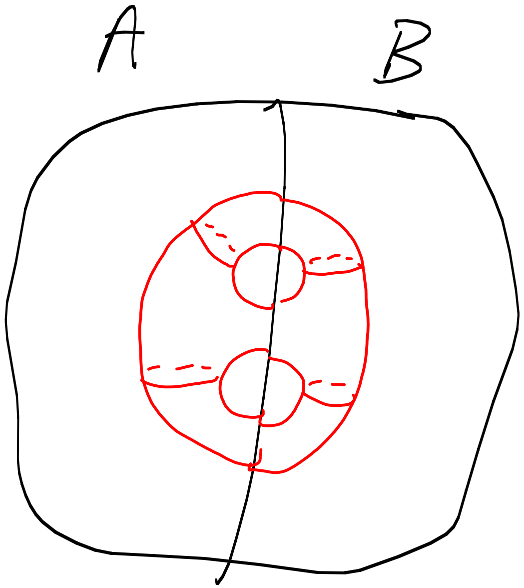
$$g_1 \sim g_2 \quad \& \quad g_1' \sim g_2' \not\Rightarrow \quad g_1 g_1' \sim g_2 g_2'$$

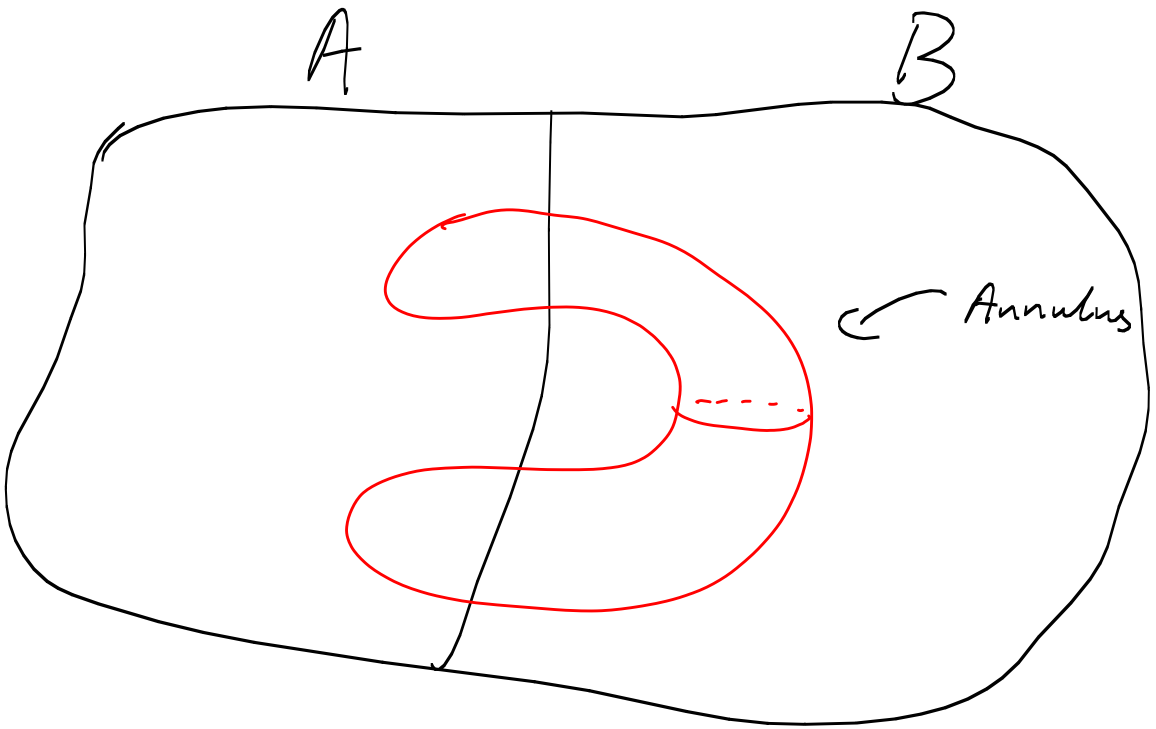
* To get a well-defined multiplication, we must *abelianise*

- $H_1 = \pi_1 / [\pi_1, \pi_1]$.

Rk: It is useful to consider e.g. $[\pi_1, \pi_1] / [\pi_1, [\pi_1, \pi_1]]$

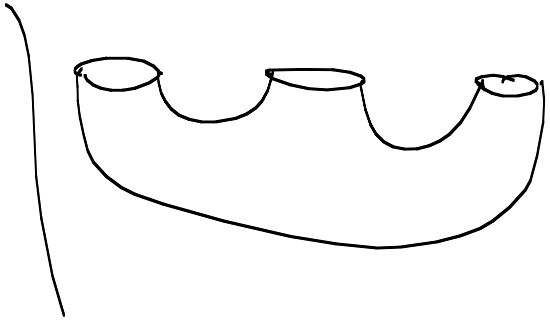
Why no combination theorems for π_2?





* Pieces in $S^2 \cap A$ & $S^2 \cap B$ are general planar surfaces

* Gluing planar surfaces gives all orientable surfaces



6/1/10

Last time
* Chains and boundaries
* Why homology is nicer.

Next:

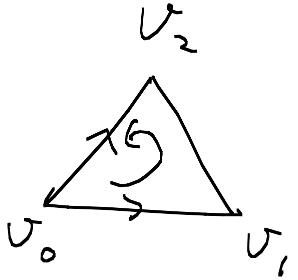
* Δ - complexes (Ref. Hatcher)
    Simplices
    Canonical linear maps
    Quotients

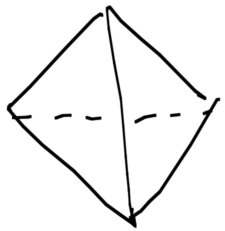
* Chain complexes
    General definition
    For Δ - complexes

* Homology

Δ - Complexes :

Simplices :





Definition: An n-simplex is the convex hull of a collection of points $v_0, v_1, \dots, v_n \in \mathbb{R}^N$ s.t. $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are independent vectors. (N some integer)

- We denote such an n-simplex as

$$\sigma = \langle v_0, v_1, \dots, v_n \rangle$$

- We assume v_i are <u>ordered</u>.
- $\sigma = \{ \sum_{i=0}^{n} a_i v_i : 0 \leq a_i \leq 1, \sum_{i=0}^{n} a_i = 1 \}$

Canonical Linear Maps:

Let $\sigma = \langle v_0, \ldots, v_n \rangle$ and $\tau = \langle w_0, \ldots, w_n \rangle$

be two n-simplices.

$$\sigma = \{ \sum_i a_i v_i : a_i \geq 0, \sum_i a_i = 1 \}$$

$$\tau = \{ \sum_i b_i w_i : b_i \geq 0, \sum_i b_i = 1 \}$$

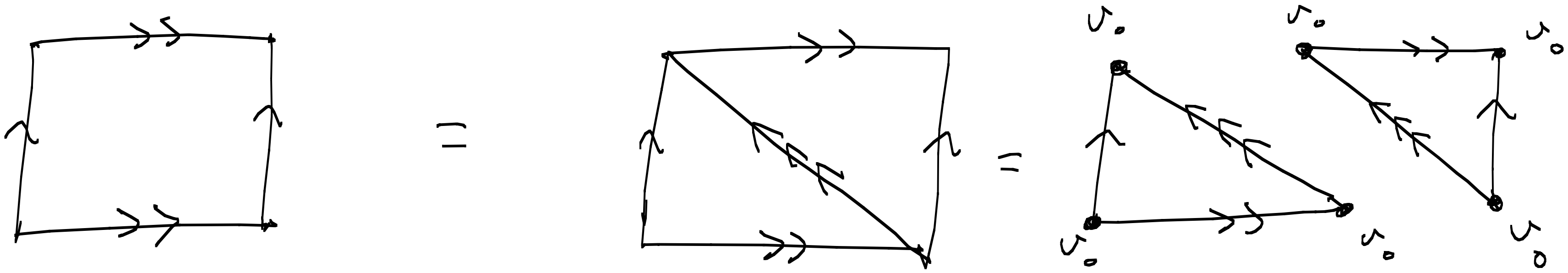
We define

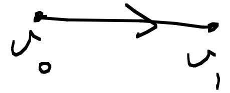
$$L : \sigma \longrightarrow \tau$$

by $L : \sum_{i=0}^{n} a_i v_i \longmapsto \sum_{i=0}^{n} a_i w_i$

Exercise: This is a bijection. (and well-defined)

Example: Δ-complex for the torus



Convention: $\langle v_0, v_1 \rangle$ is oriented 

A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

(i) The restriction $\sigma_\alpha | \mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha | \mathring{\Delta}^n$.

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Faces of a simplex:

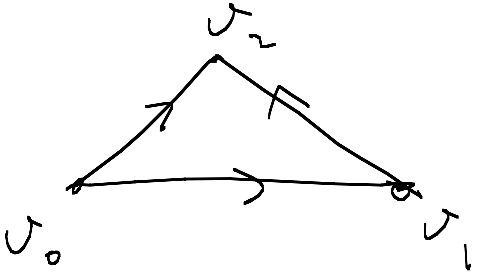
- If $\sigma = \langle v_0, \dots, v_n \rangle$, and $0 \leq i_1 < i_2 < \dots < i_k \leq n$, then $\langle v_{i_1}, v_{i_2}, \dots, v_{i_n} \rangle$ is a face of σ.

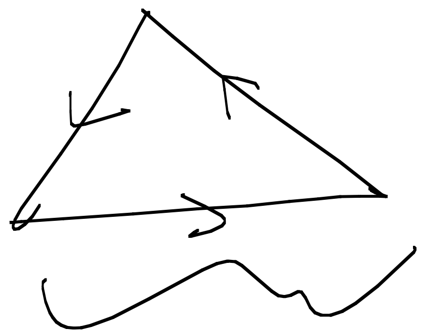
- The order of vertices in a face is the restriction of the order in a simplex.
- Hence there is a canonical linear map from a face to the simplex

$$\sum_{i=1}^{k} a_j v_{i_j} \longmapsto \sum a_j v_{i_j}.$$

Consistency of orientations

$v_0 \longrightarrow v_1 = \langle v_0, v_1 \rangle$





Not consistent.

- Suppose edges of a simplicial complex are oriented.
- This gives an ordering on each 2-simplex consistent with the orientation if no triangle has a 'cycle' as its boundary ordering.

Question: Is this sufficient for n-simplices?

The Standard Simplex:

$$\Delta^n = \{ (a_0, \dots, a_n) : a_i \geq 0 , \sum_{i=0}^{n} a_i = 1 \}$$

- Here we can take v_i to be the standard basis of $\mathbb{R}^{n+1}$.

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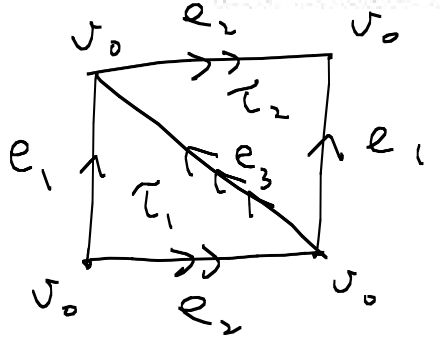


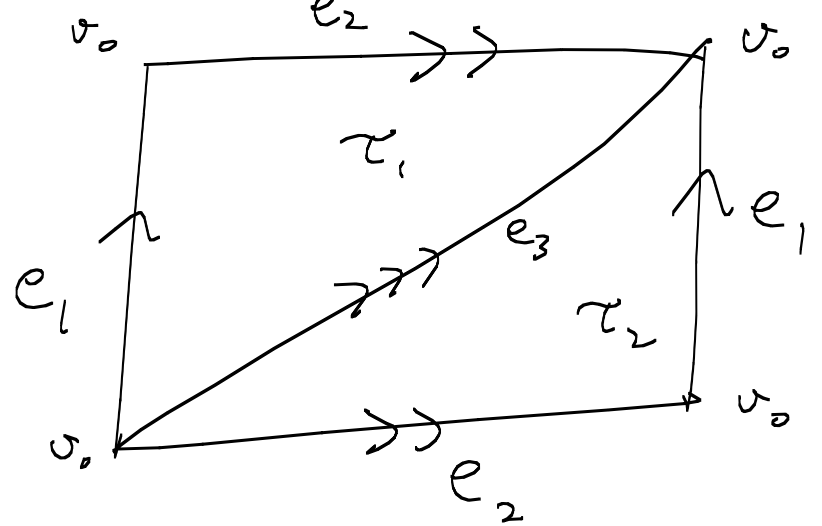
$\sigma_1 : \Delta^0 \longrightarrow T^2$

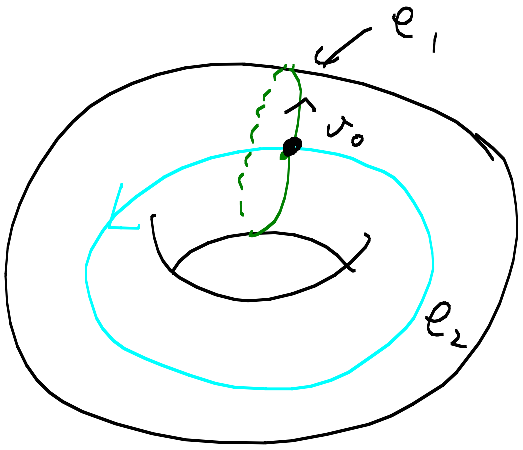
$\sigma_1 : (1) \longmapsto v_0$

$\sigma_2, \sigma_3, \sigma_4 : \Delta^1 \longrightarrow T^2$

These are the canonical maps

The interior of $\langle v_0, \dots, v_n \rangle$ is $\{ \sum_{i=0}^{n} a_i v_i : a_i > 0 , \sum_{i=1}^{n} a_i = 1 \}$





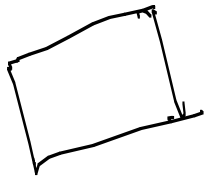
- $\sigma_1 : \Delta^0 \longrightarrow T^2$, $\sigma((1)) = v_0$
- $\sigma_2 : \Delta^1 \longrightarrow T^2$ with image e_1

- There is a canonical linear map determined by the order.
- Similarly, we have $\sigma_3, \sigma_4 : \Delta^1 \longrightarrow T^2$ with images e_2, e_3

$\sigma_5 : \Delta^2 \longrightarrow T^2$ with image τ_2

- Restriction to the boundary of τ_2 is already defined.

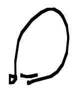
<u>Analogy</u>: A graph Γ is determined by



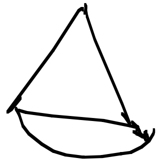
- * The vertex set $V(\Gamma)$
- * When vertices v_0, v_1 bound an edge

Provided:

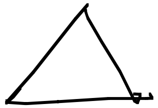
- No loops



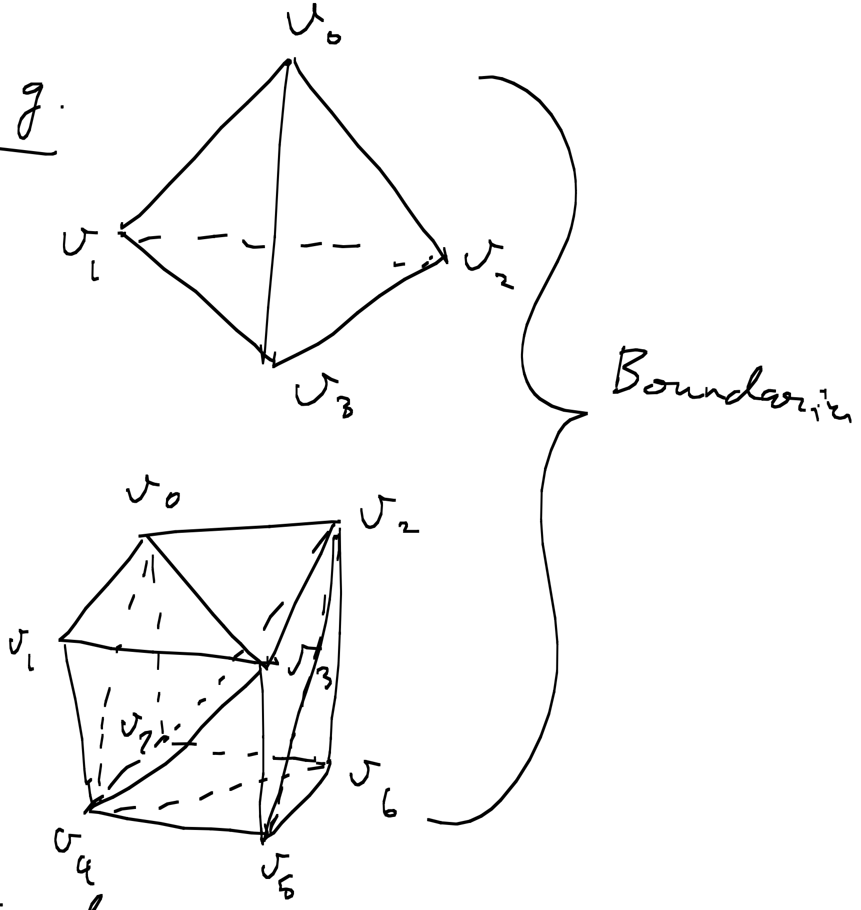
- No multiple edges.



Simplicial Complexes and their homology

- Simplicial complex is made of simplices σ.
- Here
  - * vertices •
  - * Edges 
  - Triangles 

E.g.



To specify this, we give:

- * A collection of vertices
- * Which vertices bound a simplex.

In the cube, $\langle v_0, v_1, v_2 \rangle$ is not a simplex, $\langle v_1, v_1, v_3)$ is

Definition: A simplicial complex Σ consists of

- A vertex set $V(\Sigma)$.
- A collection $S(\Sigma)$ of non-empty finite subsets of Σ.

such that

(1) If $\sigma \in S(\Sigma)$ and $\emptyset \neq \tau \subset \sigma$, then $\tau \in S(\Sigma)$.

sets of vertices

(2) Every singleton set $\{v\}$, $v \in V(\Sigma)$ is in $S(\Sigma)$.

- If $|\sigma| = k$, then σ is called a k-simplex.

E.g. **POSET complex**: Let $(V, \leq)$ be a partially ordered set.

Exercise.
- Vertices: V
- Simplices: $\sigma \in S(\Sigma)$ if $\leq$ is a total order on σ.

Geometric Realisation:

Let Σ be a simplicial complex.

We associate to it a topological space

$$X = \left\{ \underbrace{\sum_{i=0}^{n} a_i v_i}_{\text{Formal linear combination}} : \{v_0, \dots, v_n\} \in S(\Sigma), \sum_{i=0}^{n} a_i = 1, a_i \geq 0 \right\}$$

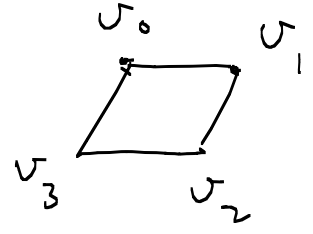
$$X = X(\Sigma) = |\Sigma|$$

$\Big\} \longrightarrow$

- Consider a vector space with basis vertices of $V(\Sigma)$
- These are elements in the. V.S.

i.e., we consider linear combinations of vertices that bound a simplex.

- We order the vertices $\{v_0, \dots, v_n\}$ in each simplex $\sigma = \{v_0, \dots, v_n\}$



'Consistency of ordering': If $\tau < \sigma$, the order on τ is the restriction of the order on σ to τ,

Propn: A consistent ordering exists.

Pf: Order $V(\Sigma)$ and restrict. $\square$

Canonical maps σ_α

- Let $\alpha = \langle v_0, \dots, v_k \rangle$ be a simplex in Σ, i.e.,
  - $\{v_0, \dots, v_k\} \in S(\Sigma)$
  - $v_0 < v_1 < \dots < v_k$ in the order on $\{v_0, \dots, v_k\}$.

There is a canonical map

$$\sigma_\alpha : \Delta^k \longrightarrow X(\Sigma) = |\Sigma|$$

given by $(a_0, \dots, a_k) \longmapsto \sum_{i=0}^{k} a_i v_i \in X(\Sigma) = |\Sigma|$

Rk: σ_α is injective. (Exercise)

Topology on $|\Sigma)$: $U \subset |\Sigma)$ is open if $\forall \alpha$ simplices of σ, $\sigma_\alpha^{-1}(U)$ is open in Δ^k.

Rk: If α is a simplex and $\beta \subset \alpha$ is a subsimplex (the order comes from α), then σ_β is the 'restriction' of σ_α,

i.e.,

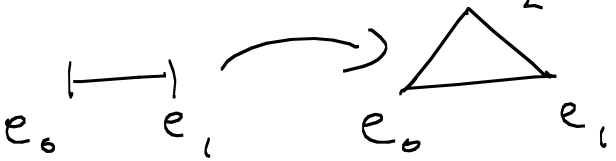
Canonical linear map

$$\begin{tikzcd}
\Delta^k \arrow[r, "\sigma_\alpha"] & {|\Sigma|} \\
\Delta^l \arrow[u, hook] \arrow[ur, "\sigma_\beta"'] &
\end{tikzcd}$$

commutes

E.g. $\alpha = \langle v_0, v_1, v_2 \rangle$, $\beta = \langle v_0, v_2 \rangle$

Canonical linear map: $\langle e_0, e_1 \rangle \longmapsto \langle e_0, e_2 \rangle \subset \langle e_0, e_1, e_2 \rangle$



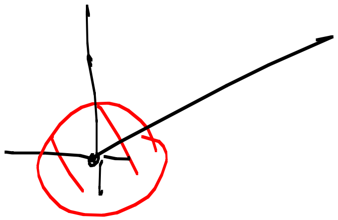
Exercise: {State and prove}

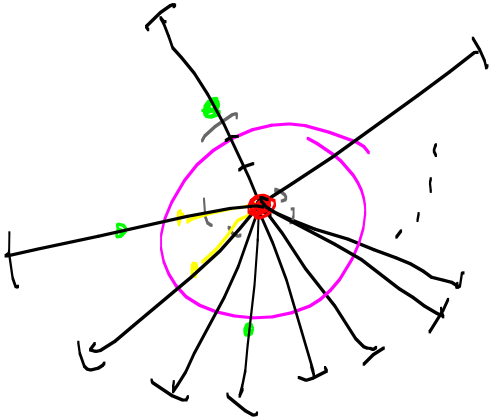
Defn: A simplicial complex Σ is locally finite if every vertex (hence every simplex) is contained in only finitely many simplices.

I_n = image of $n \times [0,1]$

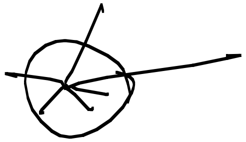
E.g. Three topologies on $X = \mathbb{N} \times [0,1]$ / $(k,0) \sim (l,0)$ $\forall k, l$.

(1) Nbds of $\{0\}$: Images of $\mathbb{N} \times [0, \delta)$, $\delta > 0$





Metric Topology



(2) $U \subset X$ open if $U \cap I_n$ is open in I_n $\forall n$.

(3) $U \subset X$ open if $U \cap I_n$ is $\begin{cases} \cdot \text{ Open } \forall n. \\ \text{or } \cdot\ I_n \text{ for all but finitely many } n, \ 0 \in U \end{cases}$

Want: Restriction to finite subcomplexes the 'obvious' topology.

Chain Complexes, Homology, Simplicial homology

Defn: A chain complex is a collection $\{C_n\}_{n\geq 0}$ of ***free*** abelian groups together with homomorphisms $\partial_n : C_n \longrightarrow C_{n-1}$, $n \geq 1$ such that $\partial_{n-1} \circ \partial_n = 0 \ \forall n \geq 2$.

Remarks: (1) In general, for R a ring we consider chain complexes (C_*, ∂_*) over R, consisting of

- Free R-modules C_k, $k \geq 0$
- R-module homomorphisms $\partial_* : C_* \longrightarrow C_{*-1}$.

such that $\partial^2 = 0$

(2) We can replace free modules by projective modules.

Remark on Free Modules etc.

(*) If F is a free abelian group (R-modules similar) and $\varphi : A \to F$ is a surjective homomorphism

$$\begin{tikzcd} A \arrow[r, "\varphi"] & F \arrow[r] \arrow[l, dashed, bend left, "\psi"] & 0 \end{tikzcd}$$

then $\exists\, \psi : F \to A$ homomorphism s.t. $\varphi \circ \psi = id_F$.

Pf: * Take a basis $x_1, \dots, x_n$ of F

* Let $y_i \in A$ be s.t. $\varphi(y_i) = x_i$.

* Define $\psi(x_i) = y_i$ $\square$

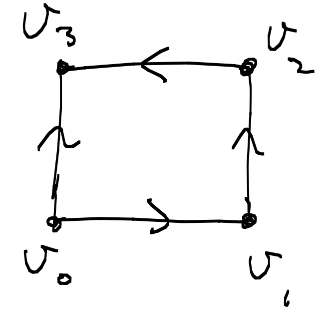
Ex: If F is a f.g. abelian group, show (*) implies F is free

- In general, an R-module is said to be projective iff it satisfies (*).

Simplicial Chain Complex:

Let Σ be a simplicial complex (with coherent orientations)

- For $k \geq 0$, $C_k(\Sigma)$ is the free abelian group with basis k-simplices in Σ.



E.g. $C_0 = \{a_0 v_0 + a_1 v_1 + a_2 v_2 + a_3 v_3 : a_i \in \mathbb{Z}\}$

- An element of C_k is called a k-chain.

- $\partial_k : C_k \longrightarrow C_{k-1}$: Enough to specify

$$\partial_k \sigma, \quad \sigma = \langle v_0, \dots, v_k \rangle \in S(\Sigma).$$

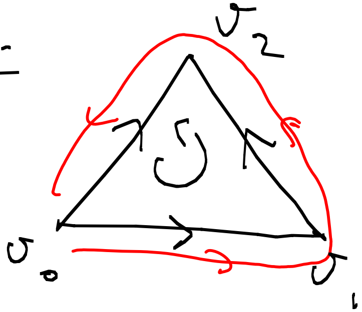
We define

$$\partial_k \underset{\substack{\uparrow \\ C_k}}{\langle v_0, \dots, v_k \rangle} = \sum_{i=0}^{k} (-1)^i \underset{\substack{\uparrow \\ C_{k-1}}}{\langle v_0, \dots, \overset{\text{Drop } v_i}{\hat{v}_i}, \dots, v_k \rangle}.$$

- As orientations are compatible, $\langle v_0, \dots, \hat{v}_i, \dots, v_k \rangle$ is a k-simplex

$$\boxed{\partial_k < v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i < v_0, \ldots, \hat{v}_i, \ldots, v_k >}$$

E.g.



$$\partial < v_0, v_1, v_2) = < v_0, v_2 > + (-1) < v_0, v_2) + < v_0, v_1$$

Theorem: $\partial_{k-1} \circ \partial_k = 0$

Pf: $\partial_k < v_0, \ldots, v_k > = \sum_{i=0}^{k} (-1)^i < v_0, \ldots, \hat{v}_i, \ldots, v_k >$

$$\partial_{k-1} \circ \partial_k < v_0, \ldots, v_k > = \sum_{i=0}^{k} (-1)^i \cdot \partial_{k-1} < v_0, \ldots, \hat{v}_i, \ldots, v_k >$$

$$= \sum_{i=0}^{k} \sum_{j<i} (-1)^i (-1)^j < v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k >$$

$$+ \sum_{i=0}^{k} \sum_{j>i} (-1)^i (-1)^{j-1} < v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j \ldots v_k >$$

$$= \sum_{i=0}^{k} \sum_{j<i} (-1)^i (-1)^j \langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k \rangle$$

$$+ \sum_{i=0}^{k} \sum_{j>i} (-1)^i (-1)^{j-1} \langle v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j \cdots v_k \rangle \longleftarrow \text{interchange } i \ \& \ j$$

$$= \sum_{i=0}^{k} \sum_{\substack{j=0 \\ j<i}}^{k} \Big[(-1)^{i+j} \langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k \rangle + (-1)^{i+j-1} \langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k \rangle$$

$$= 0$$

$$\square$$

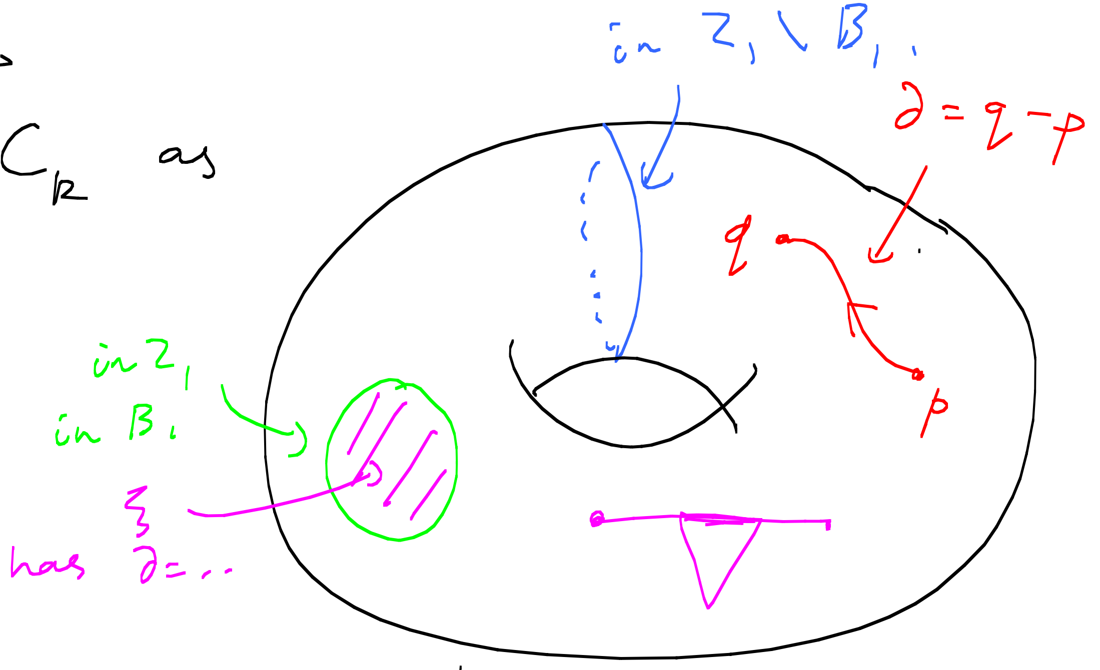
Thus, (C_k, ∂_k) is a chain complex.

Cycles and Boundaries

We define $Z_k, B_k \subset C_k$ as

$$Z_k = \ker(\partial_k),$$

$$B_k = \operatorname{Im}(\partial_{k+1})$$



Cor: $B_k \subset Z_k$

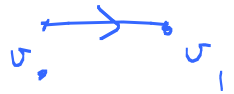
Pf: $b \in B_k \Rightarrow b = \partial_{k+1} \xi, \ \xi \in C_{k+1}$

$\Rightarrow \partial_k b = \partial_k \circ \partial_{k+1} \xi = 0$

$\Rightarrow b \in Z_k$

$\square$

. All curves represent elements of C_1.



$\partial \langle v_0, v_1 \rangle = \langle v_1 \rangle - \langle v_0 \rangle$

Homology of a chain complex

Definition: The homology groups H_k of a chain complex (C_k, ∂) are defined as

$$H_k = Z_k / B_k = \ker(\partial_k) / \operatorname{Im}(\partial_{k+1}).$$

- In particular

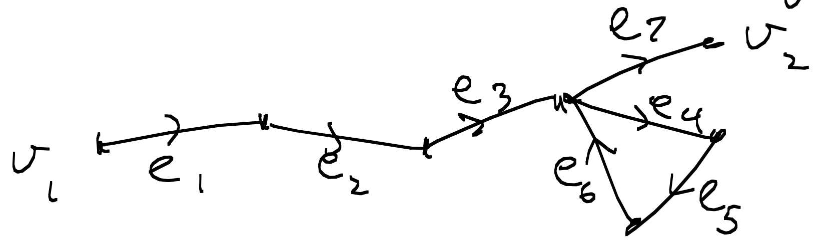
$$H_k(\Sigma) = Z_k(\Sigma) / B_k(\Sigma) \text{ for a simplicial complex } \Sigma.$$

- $b_k = \operatorname{rank}(H_k)$ is called the k-th Betti number.

Main theorem: If Σ_1 & Σ_2 are simplicial complexes such that $|\Sigma_1|$ is homeomorphic to $|\Sigma_2|$, then $H_k(\Sigma_1) \cong H_k(\Sigma_2)$ $\forall k$.

Example: $H_0(\Sigma)$, Σ a simplicial complex

- An edge path in Σ is a sequence of oriented edges $e_1, \dots, e_k$ s.t. the terminal vertex of e_i is the initial vertex of e_{i+1}



- We say vertices v_1 and v_2 are connected if there is an edge path from v_1 to v_2

- **Propn:** This gives an equivalence relation on vertices

- The equivalence classes are called the components.

Thm: $H_0(\Sigma) = \mathbb{Z}^{\text{Set of components.}}$

E.g. Σ has one component $\Rightarrow H_0(\Sigma) = \mathbb{Z}$.

- $H_0(\Sigma) = Z_0/B_0$, $Z_0 \subset C_0 = \{\sum a_i \langle v_i \rangle : v_i$ verti

- Define $\varphi : Z_0 \longrightarrow \mathbb{Z}$

$$\varphi : \sum_{i=1}^{k} a_i \langle v_i \rangle \longmapsto \sum_{i=1}^{k} a_i$$

- Lemma: $\varphi(\partial_1 \langle v_0, v_1 \rangle) = 0$

$$\rotatebox{90}{=}$$

$$\varphi(-\langle v_0 \rangle + \langle v_1 \rangle) = 0$$

Cor: $\varphi|_{B_0} \equiv 0$

- Hence we have an induced homomorphism

$$\varphi : \overset{= C_0}{Z_0}/B_0 \longrightarrow \mathbb{Z}.$$

- Lemma: This is an isomorphism.

Last time: (Blackboard lecture)

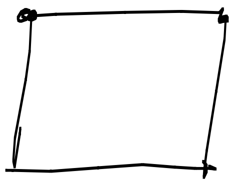
$$H_0(\Sigma) = \bigoplus_{\substack{\text{components} \\ \text{of } \Sigma}} \mathbb{Z}$$

Next:

- From simplicial complexes to Δ-complexes
- Examples of simplicial homology.

Simplicial Complex $\Sigma \longrightarrow |\Sigma|$ - topological space

E.g. $\Sigma =$



- Simplices $\alpha \in \Sigma$ correspond to subsets of $|\Sigma|$

$\|$

$\left\{ \sum_{i=0}^{k} a_i v_i : \langle v_0, \dots, v_k \rangle \right.$ is a simplex$\left.\right\}$

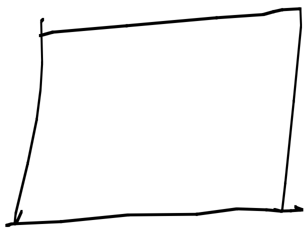
$\sum a_i = 1,\ a_i \geq 0$

Propn: Each point in $|\Sigma|$ is in the interior of a unique simplex α.

Namely, if $x = \sum_{i=0}^{k} a_i v_i$, $0 < a_i \leq 1$.

- Then $x \in \text{int}(\langle v_0, \dots, v_k \rangle)$ and not to any face.
- If $\alpha \subset \beta = \langle v_0, \dots, v_{k+1} \rangle$ (for e.g.), then $x \notin \text{int}(\beta)$.

Thus,



$$|\Sigma| = \bigcup_{\substack{\alpha \in \Sigma \\ \text{a simplex}}} \text{int}(\alpha)$$

- $\sigma_\alpha : \Delta^n \longrightarrow |\Sigma|$ are canonical $\overset{\text{linear}}{\wedge}$ maps corresponding to each $\alpha \in S(\Sigma)$.

Then

$$\boxed{|\Sigma| = \bigcup_{\alpha \text{ simplex}} \sigma_\alpha(\underbrace{\text{int}(\Delta^n)}_{\{(a_0, \dots, a_n) : 0 < a_i, \sum_{i=1}^{n} a_i = 1\}}) = \overset{o}{\Delta}{}^n}$$

- The topology on $|\Sigma|$ is given by

  U open $\Leftrightarrow$ $U \cap \alpha$ open $\forall \alpha$ $\Leftrightarrow$ $\sigma_\alpha^{-1}(U)$ open $\forall \alpha$.

- If β is a face of α, σ_β is the 'restriction' of σ_α

Δ-complexes

E.g. $\Sigma \longrightarrow |\Sigma| = X$

$\sigma_\alpha : \Delta^n \longrightarrow X$

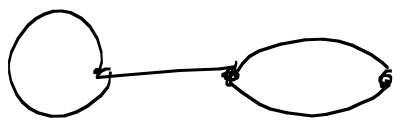
A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

(i) The restriction $\sigma_\alpha | \mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha | \mathring{\Delta}^n$.

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

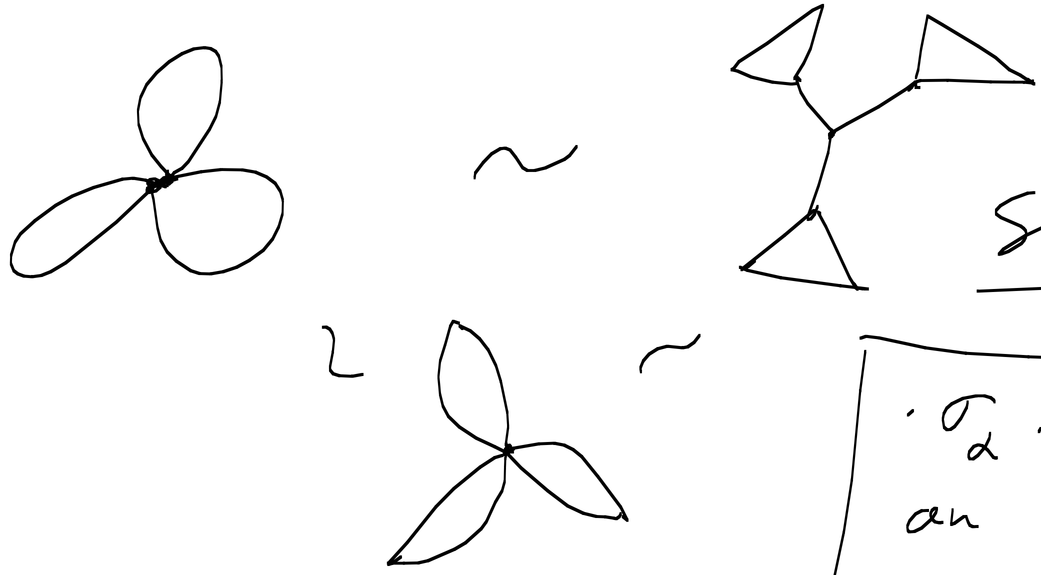
(iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

- We have dropped the requirement that $\sigma_\alpha : \Delta^n \longrightarrow X$ is injective and replaced by $\sigma_\alpha : \mathring{\Delta}^n \longrightarrow X$ is injective



- We also permit more than one simplex with the same boundary.

E.g.

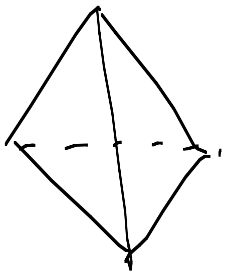


Simplicial

$\cdot\ \sigma_\alpha : \Delta^n \to X$ is called an n-simplex in X

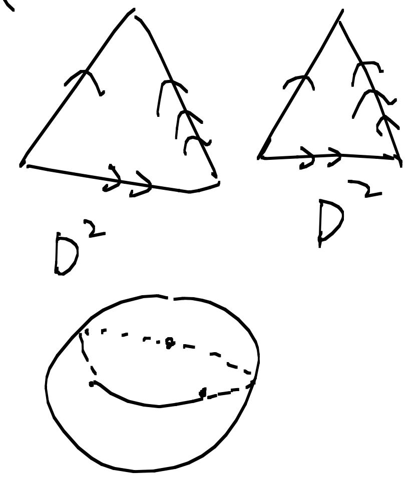
E.g. For S^2

(1)

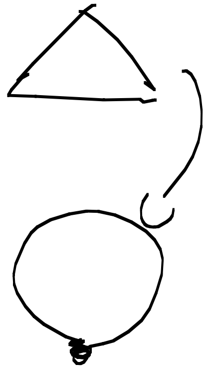


Boundary of the tetrahedron

(2) $S^2 = D^2 \cup D^2$



(3) $S^2 = D^2 / \partial D^2$



Simplicial Homology of a Δ-complex

Let $(X, \{\sigma_\alpha\}_{\alpha \in A})$ be a Δ-complex.

$$\boxed{\Delta^n = \langle e_0, \dots, e_n \rangle}$$

- An n-simplex in X is a map

$$\sigma_\alpha : \Delta^n \longrightarrow X, \qquad \alpha \in A$$

- $C_n(X) =$ free abelian group generated by n-simplices.

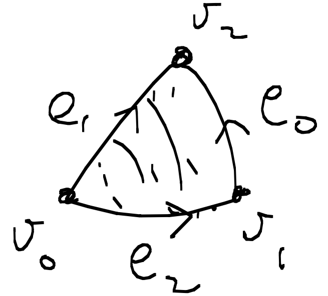
- By hypothesis, $\sigma_\alpha|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle} = \sigma_\beta$ for some unique $\beta \in A$.

Hence, we can define

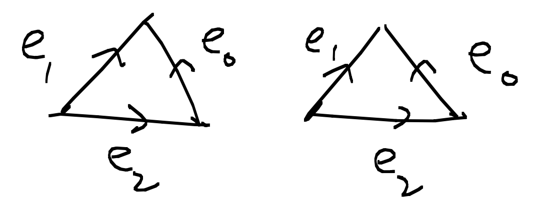
$$\partial \sigma_\alpha = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}.$$

$\boxed{\text{Ex: } \partial^2 = 0}$

Examples: $S^2 = D^2 \cup D^2$



Faces f_1, f_2



The chain complex of this Δ-complex is

$$\begin{tikzcd} C_2 = \mathbb{Z} f_1 + \mathbb{Z} f_2 \arrow[d, "\partial_2"] \\ C_1 = \mathbb{Z} e_0 + \mathbb{Z} e_1 + \mathbb{Z} e_2 \arrow[d, "\partial_1"] \\ C_0 = \mathbb{Z} v_0 + \mathbb{Z} v_1 + \mathbb{Z} v_2 \end{tikzcd}$$

$H_0 = \mathbb{Z}$

$\partial_1 e_1 = v_2 - v_0$, $\partial_1 e_2 = v_1 - v_0$,

$\partial_1 e_0 = v_2 - v_1$

$\operatorname{Im} \partial_1 = \langle v_2 - v_0, v_1 - v_0, v_2 - v_1 \rangle$

$= a_0 v_0 + a_1 v_1 + a_2 v_2$, $\sum_{i=0}^{2} a_i = 0$

<u>Ker ∂_1</u> : $0 = \partial_1 (\underbrace{a_1 e_1 + a_2 e_2}_{\xi} + a_0 e_0)$

$= (-a_1 - a_2) v_0 + (a_2 - a_0) v_1 + (a_0 + a_1) v_2$

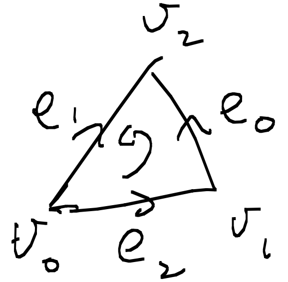
$\Leftrightarrow$ $a_2 = a_0 = -a_1$

$\Leftrightarrow$ $\xi = a(e_0 - e_1 + e_2)$

$$\begin{tikzcd} C_2 \arrow[d, "\partial_2"] \\ C_1 \ni \xi \arrow[d, "\partial_\phi"] \\ C_0 \end{tikzcd}$$

- $\xi \in \ker(\partial_1) \iff \xi \in \mathbb{Z}(e_0 - e_1 + e_2)$

  i.e. $\ker(\partial_1) = \mathbb{Z}(e_0 - e_1 + e_2)$



- $\partial_2(f_1) = e_2 + e_0 - e_1 = \partial_2(f_2)$

  $\therefore \operatorname{Im}(\partial_2) = \mathbb{Z}(e_0 - e_1 + e_2)$

  $\therefore H_1(\Sigma) = \ker(\partial_1) / \operatorname{Im}(\partial_2) = 0$

- $H_2(\Sigma) = \ker(\partial_2) = \mathbb{Z}(f_1 - f_2)$

  as $\partial_2 : a_1 f_1 + a_2 f_2 \longmapsto (a_1 + a_2)(e_0 - e_1 + e_2)$

$\therefore H_2(\Sigma) \cong \mathbb{Z}.$

For $n > 2$, $C_n = 0 \implies H_n = 0$ $\left[C_n = 0 \implies \ker(\partial_n) = 0 \implies H_n = 0 \right]$

E.g. $\underset{X}{\underset{\|}{S^2}} =$

$\left\{\begin{array}{ll} 1 & \text{0-simplex} \\ 1 & \text{2-simplex.} \end{array}\right.$

$$\begin{tikzcd} C_2 \cong \mathbb{Z} \arrow[d, "\partial_2"] \\ C_1 \cong 0 \arrow[d, "\partial_1"] \\ C_0 \cong \mathbb{Z} \end{tikzcd}$$

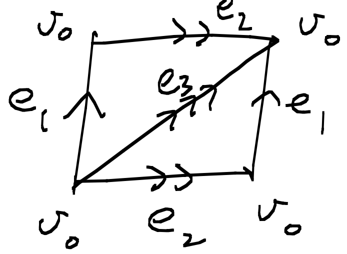
$\Rightarrow \partial_1 = 0$ & $\partial_2 = 0$

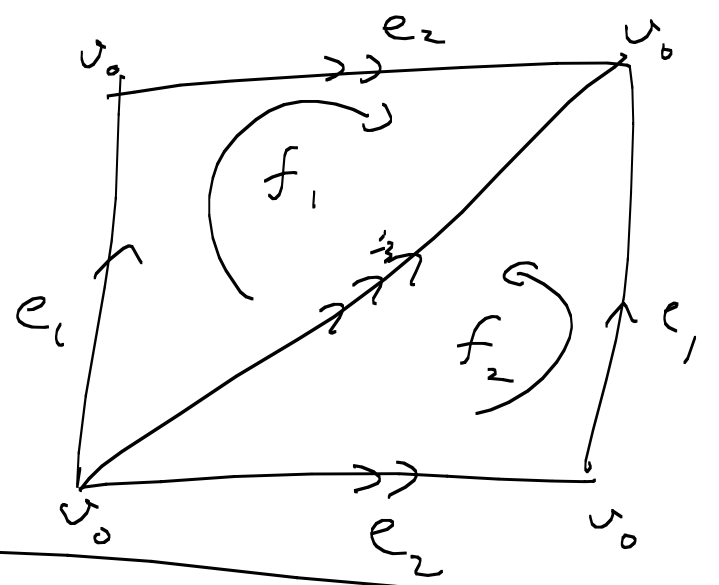
$\Rightarrow H_1 = \ker(\partial_1) / \operatorname{Im}(\partial_2) = C_1 / \operatorname{Im}(\partial_2) \cong 0$

& $H_0 = C_0 / \underset{\underset{0}{\|}}{\operatorname{Im}(\partial_1)} = C_0 = \mathbb{Z}$

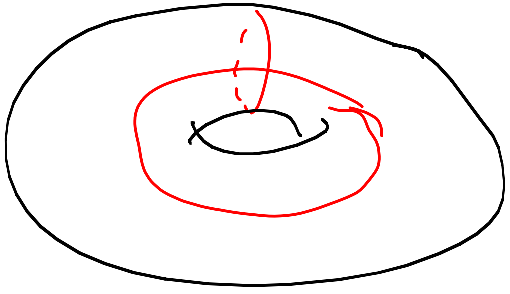
$H_2 = \ker(\partial_2) / \operatorname{Im}(\partial_3) = C_2 = \mathbb{Z}.$

Thus, if we have simplices only in even dimensions, then $H_k \cong C_k \ \forall k.$ $\underline{(\text{Ex})}$

E.g. $T^2 =$ 



$$\begin{tikzcd} C_2 = \mathbb{Z}f_1 + \mathbb{Z}f_2 \arrow[d, "\partial_2"] \\ C_1 = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 \arrow[d, "\partial_1"] \\ C_0 = \mathbb{Z}v_0 \end{tikzcd}$$



$\partial_1(e_1) = \partial_1(e_2) = \partial_{\cancel{1}}(e_3) = 0$

$\therefore \ker(\partial_1) = C_1$

$\partial_2(f_1) = e_1 + e_2 - e_3$

$\partial_2(f_2) = e_1 + e_2 - e_3$

$\therefore \operatorname{Im}(\partial_2) = \mathbb{Z}(e_1 + e_2 - e_3)$

$$H_1 = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 \big/ \mathbb{Z}(e_1 + e_2 - e_3)$$

$$\cong \mathbb{Z}^2.$$

Propn: $\underbrace{\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3}_{\mathbb{Z}^3} \Big/ \underbrace{\mathbb{Z}(e_1 + e_2 - e_3)}_{\mathbb{Z}(1,1,-1)} \cong \mathbb{Z}^2$

Pf: Let $\varphi : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2$ be

$$\varphi(e_1) = (1,0)$$
$$\varphi(e_2) = (0,1)$$
$$\varphi(e_3) = (1,1)$$

- This induces a homomorphism $\varphi : \mathbb{Z}^3 / \mathbb{Z}(1,1,-1) \to \mathbb{Z}^1$.
- This is onto.
- This is injective: $\varphi : (a,b,c) \longmapsto (a+c, b+c)$

$\therefore \varphi(a,b,c) = (0,0) \Rightarrow a = -c,\ b = -c = a.$

$\Rightarrow (a,b,c) = a(1,1,-1)$

□

Fun with Exact Sequences

Consider a sequence of R-modules & R-module homomorphisms

$$\ker(\varphi_1) = \operatorname{im}(\varphi_0)$$

$$A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} - - -$$

Defn: We say this is an exact sequence if $\forall i$, $\ker(\varphi_{i+1}) = \operatorname{im}(\varphi_i) \subset A_{i+1}$

E.g.s (1) $A \xrightarrow{\varphi} B \longrightarrow 0$ is exact iff φ is onto.

(2) $0 \longrightarrow A \longrightarrow B$ is exact iff φ is injective.

(3) $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ exact is called a short exact sequence.

(4) $0 \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$ exact $\Longleftrightarrow$ φ is morphism.

(5) $C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2 \leftarrow \cdots \leftarrow C_n \leftarrow$ is exact iff

$H_k = 0 \ \forall k \geq 1.$

Five lemma: Consider a commutative diagram with rows exact.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "\alpha"] & B \arrow[r] \arrow[d, "\beta", "\simeq"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"', "\simeq"] & E \arrow[d, hook, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

if $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Rk: We only need : $\begin{cases} \alpha & \text{is onto} \\ \varepsilon & \text{is one-to-one} \end{cases}$

P.f: (1) <u>γ is one-to-one.</u>

- Let $\gamma(x) = 0$, $x \in C$

$$\begin{tikzcd}
\overset{v}{A} \arrow[r] \arrow[d, "\alpha"] & \overset{w}{B} \arrow[r] \arrow[d, "\beta", "\simeq"'] & \overset{x}{C} \arrow[r] \arrow[d, "\gamma"'] & \overset{y}{D} \arrow[r] \arrow[d, "\delta"', "\simeq"] & E \arrow[d, "\varepsilon"] \\
\underset{v'}{A'} \arrow[r] & \underset{w'}{B'} \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

$w' \longmapsto$ $\qquad 0 \longmapsto 0$

- $x \longmapsto 0$ in C'
- Suppose $x \longmapsto y \in D$, then $y \longmapsto 0$

  $\Rightarrow y = 0$
- Thus, $x \longmapsto 0$ in $D \Rightarrow \exists\, w \in B$, $w \longmapsto x$
- $w \longmapsto w' \in B'$; $w' \longmapsto 0$ in C'.
- $\Rightarrow \exists\, v' \in A'$, $v' \longmapsto w'$.
- $\Rightarrow \exists\, v \in A$ s.t. $v \longmapsto v'$

<u>Claim:</u> $v \longmapsto w$.

Claim: $v \mapsto w \in B$

$$\begin{tikzcd}
{}^{v}A \arrow[r] \arrow[d, "\alpha"] & B^{w} \arrow[r] \arrow[d, "\beta", "\simeq"'] & C^{x} \arrow[r] \arrow[d, "\gamma"'] & D^{y} \arrow[r] \arrow[d, "\delta"', "\simeq"] & E \arrow[d, hook, "\varepsilon"'] \\
A'_{v'} \arrow[r] & B'_{w'} \arrow[r] & C'_{0} \arrow[r] & D'_{0} \arrow[r] & E'
\end{tikzcd}$$

Pf: Suppose $v \mapsto w_1$, then by commutativity, $w_1 \mapsto w'$
$\Rightarrow w = w_1$ as β is 1-1.

$$\begin{tikzcd}
v \arrow[r, mapsto] \arrow[d, mapsto] & w_1 \arrow[d, mapsto, dotted] \\
v' \arrow[r, mapsto] & w'
\end{tikzcd}$$

Now, $v \mapsto w \mapsto x \Rightarrow x = 0$ by exactness.

Thus, $\gamma(x) = 0 \Rightarrow x = 0$; γ is 1-1.

(2) γ is onto

Suppose $x' \in C'$

$$\begin{tikzcd}
 & & x_1 \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & y \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & z \\
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta", "\simeq"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"', "\simeq"] & E \arrow[d, hook, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \\
 & & x' \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & y' \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & 0
\end{tikzcd}$$

- Let $x' \longmapsto y' \in D'$
- $\exists\ y \in D$ s.t. $y \longmapsto y'$
- $y \longmapsto z \in E \longmapsto 0$ in E' $\left(\text{by} \begin{tikzcd} & y \arrow[r, mapsto] \arrow[d, mapsto] & z \arrow[d, mapsto] \\ x' \arrow[r, mapsto] & y' \arrow[r, mapsto] & 0 \end{tikzcd} \right)$

  $\Rightarrow z = 0$ (ε is 1-1)
- Thus $y \longmapsto 0 \Rightarrow \exists\ x_1 \in C$, $x_1 \longmapsto y$.
- Let $x_1 \longmapsto x_1'$ (We may not have $x_1' = x'$).

  (Rk: x_1 is not canonically determined by x')

- Let $x_1 \longmapsto x_1' \in C'$
- $x_1' \longmapsto y'$ as $\longrightarrow$ $\left\{ \begin{tikzcd} x_1 \arrow[r, mapsto] \arrow[d, mapsto] & y \arrow[d, mapsto] \\ x_1' \arrow[r, mapsto, dashed] & y' \end{tikzcd} \right.$

$$\begin{tikzcd}
& w \arrow[d, phantom, "\cap"] & x_1 \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & y \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & z \\
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta", "\simeq"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"', "\simeq"] & E \arrow[d, hook, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] \arrow[d, phantom, "\cap"] & D' \arrow[r] \arrow[d, phantom, "\cap"] & E' \\
& & x', x_1' & y' & 0 \\
& w' \arrow[r, mapsto] & x_1' - x' & &
\end{tikzcd}$$

$\Longrightarrow x_1' - x' \longmapsto 0$

$\Longrightarrow \exists\, w' \in B',\ w' \longmapsto x_1' - x'$

- $\exists\, w \in B,\ w \longmapsto w'$
- Let $w \longmapsto x_2 \in C,\ x_2 \longmapsto x_1' - x'$

$\therefore x_1 - x_2 \longmapsto x_1' - (x_1' - x') = x'$ ☺

Thus, γ is onto.

Cor: Given a commutative diagram with rows exact

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\psi"'] & B \arrow[r] \arrow[d, "\varphi"'] & C \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

[i.e.

$$\begin{tikzcd}
 & & B \arrow[dd, dashed] \arrow[dr] & & \\
0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\
 & & B' \arrow[ur] & &
\end{tikzcd}$$

]

if ψ & η are isomorphisms, so is φ.

E.g. We have two s.e.s.'s

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r, "q"] \arrow[d, dashed] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

and (the second row above)

with the outer terms isomorphic, but there is no commutative diagram extending these.

[**Reason:**

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & 2 \arrow[d, dashed] \\
1 \arrow[r, mapsto] & (1,0)
\end{tikzcd}$$

, $(1,0)$ is not divisible by 2]

Singular Homology (Axioms determining)

Functorality:

$C_n(X,A) = C_n(X)/C_n(A)$

homology

- We associate to
  - pairs of spaces $(X,A) \longrightarrow H_n(X,A)$, $n \geq 0$
    Abelian groups

    $(X,\emptyset) \longrightarrow H_n(X)$

- Maps $f : (X,A) \longrightarrow (Y,B)$ (i.e. $f : X \to Y$, $f(A) \subset B$)

  homomorphism $f_* : H_n(X,A) \longrightarrow H_n(Y,B)$ $\forall n \geq 0$

  s.t.
  - $id_* : H_*(X,A) \circlearrowleft$ is the identity
  - $(f \circ g)_* = f_* \circ g_*$.

i.e. Homology is a functor from pairs of spaces to graded abelian groups, $(X,A) \longrightarrow \bigoplus_{n \geq 0}^{\infty} H_n(X,A)$.

Axioms:
(1) Exactness
(2) Excision
(3) Homotopy
(4) Dimension – Homology of a point.
(5) Compact Support.

Homotopy Axiom (and a consequence)

If $f, g : (X, A) \longrightarrow (Y, B)$ are homotopic, then $f_* = g_*$ on homology

Cor: If $f : (X, A) \longrightarrow (Y, B)$ is a homotopy equivalence, then $f_* : H_*(X, A) \longrightarrow H_*(Y, B)$ is an isomorphism.

Pf: If g is the 'inverse', $f \circ g \sim id$ & $g \circ f \sim id$, then $\underset{\underset{f_* \circ g_*}{\|}}{(f \circ g)_*} = id_* = id$ & $\underset{\underset{g_* \circ f_*}{\|}}{(g \circ f)_*} = id$

Exactness Axiom: Given (X, A), there is a long exact sequence

$$H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \underset{\substack{\uparrow \\ \text{inclusion}}}{\xrightarrow{i_*}} H_n(X) \underset{\substack{\uparrow \\ \text{inclusion}}}{\xrightarrow{j_*}} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

- The homomorphisms $H_n(A) \longrightarrow H_n(X)$ and $H_n(X) \longrightarrow H_n(X, A)$ are induced by inclusion.
- Part of the statement is that homomorphisms $\partial : H_n(X, A) \longrightarrow H_{n-1}(A)$ exist.
- This homorphism ∂ is 'natural', i.e,

For $f : (X, A) \to (Y, B)$,

$$\begin{tikzcd} H_n(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "f_*"] \\ H_n(Y, B) \arrow[r, "\partial"] & H_{n-1}(B) \end{tikzcd}$$

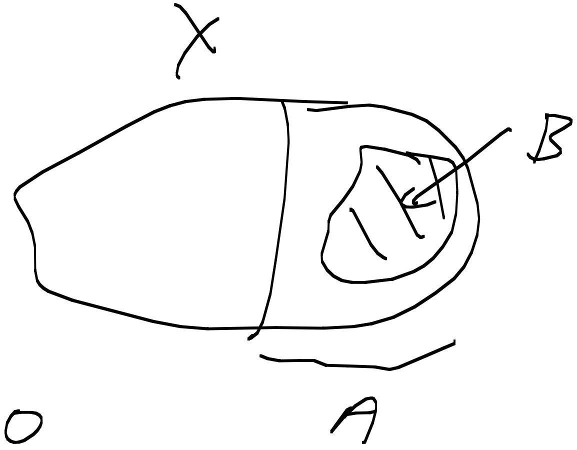
commutes,

Excision Axiom: We can excise in the interior of A without affecting $H_*(X, A)$

Axiom: If $B \subset A \subset X$ is such that $\bar{B} \subset \mathring{A}$, then the homomorphism induced by inclusion

$$i_*: H_n(X \setminus B, A \setminus B) \xrightarrow{\simeq} H_n(X, A)$$

is an isomorphism.



Dimension Axiom: $H_n(\{p\}) = \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n>0. \end{cases}$

Singular homology:

- X a topological space.

Defn: . A singular n-simplex σ_n in X is a map $\sigma_n : \Delta^n \longrightarrow X$.

- A singular n-chain $\xi \in C_n(X)$ is a formal linear combinatio of singular n-simplices

$$\xi \in C_n(X) = \left\{ \sum_{i=1}^{k} a_i \sigma_i ; a_i \in \mathbb{Z}, \underset{\text{maps}}{\sigma_i : \Delta^n \to X} \right\}$$

- $C_n(X)$ is the free abelian group with basis the singular n-simplices.

The boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$

- $\Delta^n = \langle e_0, \dots, e_n \rangle$, $\quad e_i = (0, \dots, \underset{\uparrow\, i^{th} \text{ position.}}{1}, \dots, 0)$
- There is a canonical linear map

$$\Delta^{n-1} \longrightarrow \langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle$$

for $i = 0, \dots, n$. This is a homeomorphism.

- Thus $\sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}$ corresponds to

a canonical map

$$\begin{tikzcd} \tau : \Delta^{n-1} \arrow[r] \arrow[rd, "\cong"'] & X \\ & \langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle \arrow[u, "\sigma"'] \end{tikzcd}$$

Defn: $\displaystyle \partial \sigma = \sum_{i=0}^{n} (-1)^i \sigma \Big|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle} \in C_{n-1}(X)$

$\partial : C_n(X) \longrightarrow C_{n-1}(X)$ – the linear extension.

Exercise: $\partial \circ \partial = 0$.

Thus, $(C_*(X), \partial_*)$ is a chain complex

. Its homology $H_n(X)$ is called the <u>singular homology</u> of X.

<u>Functoriality</u> : Space $X \rightsquigarrow C_*(X) \rightsquigarrow H_*(X)$

$$\text{Map } f : X \to Y \rightsquigarrow \underbrace{f_\# : C_*(X) \to C_*(Y)}_{\text{chain homomorphism}} \rightsquigarrow \underbrace{f_* : H_*(X) \to H_*(Y)}_{\text{homomorphism}}$$

- The map $f_\# : \overset{\in \sigma : \Delta^n \to X}{C_n(X)} \longrightarrow C_n(Y)$ is given by

$$f_\# : \sigma \longmapsto f \circ \sigma$$

$f : X \to Y$; $f_{\#} : \sigma \mapsto f \circ \sigma$; $f_{\#} : C_*(X) \to C_*(Y)$

Defn: If (C_*, ∂_*) & (C'_*, ∂'_*) are chain complexes, a chain homomorphism $\Phi_* : (C_*, \partial_*) \to (C'_*, \partial'_*)$ is a collection of homomorphisms

$$\Phi_n : C_n \to C'_n , \quad n \geq 0$$

s.t. the diagram

$$\begin{tikzcd} C_n \arrow[r, "\Phi_n"] \arrow[d, "\partial_n"'] & C'_n \arrow[d, "\partial'_n"] \\ C_{n-1} \arrow[r, "\Phi_{n-1}"] & C'_{n-1} \end{tikzcd}$$

commutes $\forall n \geq 1$.

Lemma A: $f_{\#}$ is a chain homomorphism.

Lemma B: A chain homomorphism Φ induces Φ on homology 'functorially'.

Lemma A: $f_\# : C_*(X) \longrightarrow C_*(Y)$ is a chain homomorphism

Pf: Enough to verify that for $\sigma : \Delta^n \to X$,

$$f_\#(\partial_n^X \sigma) = \partial_n^Y (f_\# \sigma)$$

$$\begin{tikzcd} C_n(X) \arrow[r] \arrow[d, "\partial_n^X"] & C_n(Y) \arrow[d, "\partial_n^Y"] \\ C_{n-1}(X) \arrow[r] & C_{n-1}(Y) \end{tikzcd}$$

$$\cdot \quad \partial_n^X \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}$$

$$\Rightarrow f_\# \partial_n^X \sigma = \sum_{i=0}^{n} (-1)^i f_\#\left(\sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}\right)$$

$$= \sum_{i=0}^{n} (-1)^i f \circ \left(\sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}\right)$$

$\|$ Exercise

$$= \sum_{i=0}^{n} (-1)^i \underset{\underset{f_\#\sigma}{\|}}{(f \circ \sigma)}\Big|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}$$

$$= \partial_n^Y (f_\# \sigma)$$

$\square$

Lemma B: A chain homorphism $\Phi_{\#} : (C_*, \partial_*) \to (C'_*, \partial'_*)$ induces homomorphisms $\Phi_* : H_* \longrightarrow H'_*$ between the corresponding homology groups

$$H_n = \frac{\ker(\partial_n) \subset C_n}{\operatorname{Im}(\partial_{n+1}) \subset C_n} \longrightarrow H'_n = \frac{\ker(\partial'_n) \subset C'_n}{\operatorname{Im}(\partial'_{n+1}) \subset C'_n}$$

Ex $\{$Furthermore, $id_* = id$ & $(\Phi_{\#} \circ \Psi_{\#})_* = \Phi_* \circ \Psi_*$

Pf: If $\xi \in \ker(\partial_n)$, then $\partial'_n(\Phi(\xi)) = \Phi(\partial_n \xi) = 0$

$\therefore$ $\Phi_n(\ker(\partial_n)) \subset \ker(\partial'_n)$, i.e. $\Phi_n|_{\ker(\partial_n)} : \ker(\partial_n) \to \ker(\partial'_n)$

· If $\xi \in \operatorname{Im}(\partial_{n+1})$, $\exists\, \zeta \in C_{n+1}$ s.t. $\xi = \partial \zeta$

$$\Rightarrow \Phi_n(\underset{\substack{\uparrow \\ C_n}}{\xi}) = \Phi_n(\partial_{n+1}\zeta) = \partial'_{n+1} \Phi_{n+1}(\zeta) \Rightarrow \Phi_n(\xi) \in im(\partial'_{n+1})$$

as $\Phi_n \circ \partial_{n+1} = \partial'_{n+1} \circ \Phi_{n+1}$

Thus, we have an induced homorphism.

Thus $f : X \to Y$ gives $f_{\#} : C_*(X) \to C_*(Y)$ gives $f_* : H_*(X) \to H_*(Y)$

· $(f_{\#} \circ g_{\#}) = (f \circ g)_{\#}$ - obvious ; $id_{\#} = id.$

Hence $(f \circ g)_* = f_* \circ g_*$

& $(id)_* = id$

Thus homology is a 'functor' from topological spaces to graded abelian groups.

Relative Homology: If $A \subset X$, $C_n(A) \subset C_n(X)$

$$C_n(X, A) = C_n(X) / C_n(A)$$

· ∂_n induces $\partial_n : C_n(X, A) \longrightarrow C_{n-1}(X, A)$ making this a chain complex.

Dimension Axiom: Homology of $X = \{p\}$

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- Singular n-simplices: $\sigma_n : \Delta^n \longrightarrow X$ – unique, constant map.

$\Rightarrow C_n(X) \cong \mathbb{Z} = \mathbb{Z}[\sigma_n]$

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] \arrow[d, phantom, "\cong" sloped] & C_2 \arrow[r, "\partial_2"] \arrow[d, phantom, "\cong" sloped] & C_1 \arrow[r, "\partial_1"] \arrow[d, phantom, "\cong" sloped] & C_0 \arrow[d, phantom, "\cong" sloped] \\
\cdots \arrow[r, "id"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "id"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \\
 & & \sigma_2 & \sigma_1 &
\end{tikzcd}$$

- $\partial_n \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n|_{\langle e_0, \ldots, \hat{e}_i, \ldots, e_n \rangle}$

$$= \left(\sum_{i=0}^{n} (-1)^i \right) \sigma_{n-1}$$

$$= \begin{cases} \sigma_{n-1}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

Thus, the chain complex is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow[\partial_4]{id} \mathbb{Z} \xrightarrow[\partial_3]{0} \mathbb{Z} \xrightarrow[\partial_2]{id} \underset{C_1}{\underset{\shortparallel}{\mathbb{Z}}} \xrightarrow[\partial_1]{0} \underset{C_0}{\underset{\shortparallel}{\mathbb{Z}}}$$

- $H_0(X) = C_0(X) / \mathrm{Im}(\partial_1) = \mathbb{Z}$
- If k is odd,

$$H_k(X) = \frac{\ker(\overset{=0}{\partial_k})}{\mathrm{Im}(\underset{\overset{\shortparallel}{id}}{\partial_{k+1}})} = \overset{C_k}{\overset{\shortparallel}{\mathbb{Z}}} / \mathbb{Z} = 0.$$

- If k is even,

$$H_k(X) = \frac{\ker(\overset{=id}{\partial_k})}{\mathrm{Im}(\partial_{k+1})} = \frac{0}{0} = 0$$

Thus,

$$H_k(\{pt\}) = \begin{cases} \mathbb{Z}, k=0 \\ 0, k>0 \end{cases}$$

Relative homology and Exactness:

- Let $A \subset X$, A, X topological spaces.
- Any singular simplex in A, $\sigma : \Delta^n \to A$, is a singular simplex in X.
- Hence:
  - $C_n(A) \subset C_n(X)$
  - The basis of $C_n(A)$ is a subset of the basis of $C_n(X)$.
  - This implies that

$$C_n(X, A) := C_n(X) / C_n(A)$$

is a free abelian group with basis

$$\{\sigma : \Delta^n \longrightarrow X : \sigma(\Delta^n) \not\subset A\}$$

We have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r] & C_n(X) \arrow[r] & C_n(X,A) \arrow[r] \arrow[d, equal] & 0 \\ & & & C_n(X)/C_n(A) & \end{tikzcd}$$

for all n.

. This says: • $C_n(A) \hookrightarrow C_n(X)$

• $C_n(X,A) = C_n(X)/C_n(A)$ (Noether isomorphism theorem)

· We have seen

$$\begin{tikzcd} C_n(A) \arrow[r] \arrow[d, "\partial^A"] & C_n(X) \arrow[d, "\partial^X"] \\ C_{n-1}(A) \arrow[r] & C_{n-1}(X) \end{tikzcd}$$

commutes. (Apply 'Lemma A')

<u>Lemma:</u> ∂^X induces a homomorphism $\partial_n : C_n(X,A) \to C_{n-1}(X,A)$

In fact,

Lemma: There is a unique homomorphism

$$\partial_n^{(X,A)} : C_n(X,A) \longrightarrow C_{n-1}(X,A)$$

such that

$$\begin{tikzcd}
 & \zeta & \tilde{\xi}', \tilde{\xi} & \xi & \\
0 \arrow[r] & C_n(A) \arrow[r, "i_n"] \arrow[d, "\partial_n^A"] & C_n(X) \arrow[u, phantom, sloped, "\ni"] \arrow[r, "q_n"] \arrow[d, "\partial_n^X"] & C_n(X,A) \arrow[u, phantom, sloped, "\ni"] \arrow[r] \arrow[d, "\partial_n^{(X,A)}"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_{n-1}"] & C_{n-1}(X) \arrow[r, "q_{n-1}"'] & C_{n-1}(X,A) \arrow[r] & 0 \\
 & \partial_n^A(\zeta) & \partial_n^X \tilde{\xi} & &
\end{tikzcd}$$

Ex: Show $\partial_n^{(X,A)}$ is a homomorphism

commutes.

Pf: Given $\xi \in C_n(X,A)$, $\exists\, \tilde{\xi} \in C_n(X)$ s.t. $q_n : \tilde{\xi} \mapsto \xi$

Let $\partial_n^{(X,A)}(\xi) = q_{n-1}(\partial_n^X \tilde{\xi})$ [This is forced by commutativity] $\Downarrow$ uniqueness

Well-defined: $q_n(\tilde{\xi}') = \xi = q_n(\tilde{\xi}) \Rightarrow q_n(\tilde{\xi}' - \tilde{\xi}) = 0$

$\Rightarrow \tilde{\xi}' - \tilde{\xi} = i_n(\zeta) \Rightarrow \partial_n^X(\tilde{\xi}') - \partial_n^X(\tilde{\xi}) = i_{n-1}(\partial_n^A(\zeta))$

$\Rightarrow q_{n-1}(\partial_n^X \tilde{\xi}') - q_{n-1}(\partial_n^X \tilde{\xi}) = q_{n-1} i_{n-1}(\partial_n^A \zeta) = 0$

<u>Corollaries;</u> (1) $q_* : C_*(X) \longrightarrow C_*(X,A)$ is a chain homomorphism

$$\begin{tikzcd}
 & \xi & \tilde{\xi}', \tilde{\xi} & \xi \arrow[l, mapsto] & \\
0 \arrow[r] & C_n(A) \arrow[r, "i_n"] \arrow[d, "\partial_n^A"] & C_n(X) \arrow[r, "q_n"] \arrow[d, "\partial_n^X"] & C_n(X,A) \arrow[r] \arrow[d, "\partial_n^{(X,A)}"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_{n-1}"] & C_{n-1}(X) \arrow[r, "q_{n-1}"'] & C_{n-1}(X,A) \arrow[r] & 0 \\
 & \partial_n^A(\xi) & \partial_n^X \tilde{\xi} & &
\end{tikzcd}$$

(2) $\partial_{n-1}^{(X,A)} \circ \partial_n^{(X,A)} = 0$

<u>Pf:</u>

$$\begin{tikzcd}
\tilde{\xi} \arrow[r, mapsto] & \xi & \\
C_n(X) \arrow[r] \arrow[d, "\partial_n^X"'] & C_n(X,A) \arrow[r] \arrow[d, "\partial_n^{(X,A)}"] & 0 \\
C_{n-1}(X) \arrow[r] \arrow[d, "\partial_{n-1}^X"'] & C_{n-1}(X,A) \arrow[r] \arrow[d, "\partial_{n-1}^{(X,A)}"] & 0 \\
C_{n-2}(X) \arrow[r] & C_{n-2}(X,A) \arrow[r] & 0 \\
0 \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

- $\partial_{n-1}^X \circ \partial_n^X = 0$

$\Longrightarrow \partial_{n-1}^{(X,A)} \circ \partial_n^{(X,A)}(\xi) = 0.$

Thus, $C_*(X,A)$ is a chain complex.

Defn: $H_*(X,A)$ is the homology of $C_*(X,A)$

By the above, we have a short exact sequence of chain complexes

$$0 \to C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X,A) \longrightarrow 0,$$

i.e.,

- We have chain homomorphisms.
- The sequences for each n are exact.

Theorem: (Zig-Zag Lemma)

A short exact sequence (s.e.s) of chain complexes

$$0 \longrightarrow C'_* \longrightarrow C_* \longrightarrow C''_* \longrightarrow 0$$

induces a long exact sequence of homology groups

$$\cdots \longrightarrow H''_{n+1} \xrightarrow{\partial} H'_n \longrightarrow H_n \longrightarrow H''_n \xrightarrow{\partial} H'_{n-1} \longrightarrow H_{n-1} \longrightarrow \cdots.$$

Further, the connecting homomorphism ∂ is natural, i.e., given a commuting diagram of s.e.s. of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C'_* \arrow[r] \arrow[d] & C_* \arrow[r] \arrow[d] & C''_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{C}'_* \arrow[r] & \hat{C}_* \arrow[r] & \hat{C}''_* \arrow[r] & 0
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
H_n \arrow[r] \arrow[d] & H''_n \arrow[r, "\partial"] \arrow[d] & H'_{n-1} \arrow[r] \arrow[d] & H_{n-1} \arrow[d] \\
\hat{H}_n \arrow[r] & \hat{H}''_n \arrow[r, "\partial"] & \hat{H}'_{n-1} \arrow[r] & \hat{H}_{n-1}
\end{tikzcd}$$

commutes.

Pf: (1) Exactness at H_n for

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$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r] \arrow[d] & \overset{\tilde{\xi}}{C_{n+1}} \arrow[r] \arrow[d] & \overset{\tilde{\xi}''}{C''_{n+1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r] & C_{n-1} \arrow[r] & C''_{n-1} \arrow[r] & 0
\end{tikzcd}$$

ξ' (at C'_n), $\xi, \partial\tilde{\xi}$ (at C_n), ξ'' (at C''_n); $\partial\xi'$ (at C'_{n-1}); $\xi \mapsto 0$ (in C_{n-1})

gives

$$[\xi'] \longmapsto [\xi] \longmapsto 0$$

$$\longrightarrow H_{n+1} \xrightarrow{j_*} H''_{n+1} \xrightarrow{\partial} H'_n \xrightarrow{i_*} H_n \xrightarrow{j_*} H''_n \xrightarrow{\partial} H'_{n-1} \xrightarrow{i_*} H_{n-1}$$

- $j_* \circ i_* = 0$ follows from the exactness for C'_n, C_n, C''_n
- Suppose $j_*([\xi]) = 0$, i.e. $[\overset{=\xi''}{j_\#(\xi)}] = 0 \Rightarrow \xi'' = \partial\tilde{\xi}''$
- $\exists\, \tilde{\xi} \mapsto \tilde{\xi}''$; $\xi - \partial\tilde{\xi} \mapsto 0$; $[\partial\tilde{\xi}] = \partial[\tilde{\xi}''] = 0$; $[\xi - \partial\tilde{\xi}] = [\xi]$
- $\exists\, \xi' \mapsto \xi - \partial\tilde{\xi}$; $\partial\xi' = 0$, Hence $[\xi'] \mapsto [\xi - \partial\tilde{\xi}] = [\xi]$ $\square$

(2) <u>Construction of $\partial : H_n'' \to H_{n-1}'$</u>

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}' \arrow[r] \arrow[dd] & C_{n+1} \arrow[r] \arrow[dd] & C_{n+1}'' \arrow[r] \arrow[dd] & 0 \\
 & & \xi \arrow[r, mapsto] & \xi'' & \\
0 \arrow[r] & C_n' \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_n'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}' \arrow[r] & C_{n-1} \arrow[r] & C_{n-1}'' \arrow[r] & 0 \\
 & \xi' & \partial\xi \arrow[r, mapsto] & 0 & \\
 & \partial\xi' \arrow[r, mapsto] & \partial\partial\xi = 0 & &
\end{tikzcd}$$

gives

$$\to H_{n+1} \xrightarrow{j_*} H_{n+1}'' \xrightarrow{\partial} H_n' \xrightarrow{i_*} H_n \xrightarrow{j_*} \overset{[\xi'']}{H_n''} \xrightarrow{\partial} H_{n-1}' \xrightarrow{i_*} H_{n-1}$$

(a) <u>Definition:</u> $\exists\, \xi \mapsto \xi''$; Consider $\partial\xi$, $\partial\xi \mapsto 0$

$\Rightarrow \exists\, \xi' \mapsto \partial\xi$; $\partial\xi' = 0$ as $\partial\xi' \mapsto 0$ in C_{n-1}

Thus, we can define $\partial[\xi''] = [\xi']$.

(b) ∂ well-defined: Suppose $[\xi''] = [\alpha'']$, to show $[\alpha'] = [\xi']$
& $\alpha \longmapsto \alpha''$

$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r] \arrow[d] & \overset{\theta}{C_{n+1}} \arrow[r] \arrow[d] & \overset{\theta''}{C''_{n+1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\beta}{C'_n} \arrow[r] \arrow[d] & \overset{\partial\theta,\ \alpha,\ \xi}{C_n} \arrow[r] \arrow[d] & \overset{\alpha'',\ \xi''}{C''_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underset{\partial\beta \quad \alpha',\ \xi'}{C'_{n-1}} \arrow[r] & \underset{\partial\alpha \quad \partial\xi}{C_{n-1}} \arrow[r] & C''_{n-1} \arrow[r] & 0
\end{tikzcd}$$

gives

$$\longrightarrow H_{n+1} \xrightarrow{j_*} H''_{n+1} \xrightarrow{\partial} H'_n \xrightarrow{i_*} H_n \xrightarrow{j_*} \overset{[\alpha'']=[\xi'']}{H''_n} \xrightarrow{\partial} \overset{[\xi']}{H'_{n-1}} \xrightarrow{i_*} H_{n-1}$$

- $\alpha - \xi \longmapsto \alpha'' - \xi'' \overset{\partial}{\longmapsto} 0 \Rightarrow \exists\, \theta'',\ \partial\theta'' = \alpha'' - \xi''$
- $\partial\theta \longmapsto \alpha'' - \xi''$ in $C''_n \Rightarrow \alpha - \xi - \partial\theta \longmapsto 0$

$\Rightarrow \exists\, \beta \in C'_n,\ \beta \longmapsto \alpha - \xi - \partial\theta$

- $\partial\beta \longmapsto \partial(\alpha - \xi - \partial\theta) = \partial\alpha - \partial\xi$; $\alpha' - \xi' \longmapsto \partial\alpha - \partial\xi$
- By injectivity, $\partial\beta = \alpha' - \xi' \Rightarrow [\alpha'] = [\xi']$. $\square$

(3) Exactness at H_n''

$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & C''_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_n \arrow[r] \arrow[d] & C_n \ \xi \arrow[r] \arrow[d] & C''_n \ \xi'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underset{\xi'}{C'_{n-1}} \arrow[r] & \underset{\partial\xi}{C_{n-1}} \arrow[r] & C''_{n-1} \arrow[r] & 0
\end{tikzcd}$$

gives

$$\longrightarrow H_{n+1} \xrightarrow{j_*} H''_{n+1} \xrightarrow{\partial} H'_n \xrightarrow{i_*} \overset{[\xi]}{H_n} \xrightarrow{j_*} \overset{[\xi'']}{H''_n} \xrightarrow{\partial} H'_{n-1} \xrightarrow{i_*} H_{n-1}$$

(a) · If $[\xi] \longmapsto [\xi'']$, w.l.g. $\xi \longmapsto \xi''$ (take $\xi'' = j_\# \xi$)

· $\partial\xi = 0 \Rightarrow \xi' = 0 \Rightarrow [\xi'] = 0$. Thus, $[\xi] \longmapsto [\xi''] \longmapsto 0$.

(b) Now, suppose $\partial \xi'' = 0$, we should show $\exists\, \hat{\xi} \in C_n$ cycle s.t. $[\hat{\xi}] \longmapsto [\xi'']$

(h) <u>To show</u>: $[\xi''] \mapsto 0 \implies \exists\, \hat{\xi},\ [\hat{\xi}] \longmapsto [\xi'']$

$\cdot\ [\xi''] \overset{\partial}{\longmapsto} [\xi'] = 0$

$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r] \arrow[dd] & C_{n+1} \arrow[r] \arrow[dd] & C''_{n+1} \arrow[r] \arrow[dd] & 0 \\
& \theta' \arrow[r, mapsto] & \theta \quad \xi & \xi'' & \\
0 \arrow[r] & C'_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r] & C_{n-1} \arrow[r] & C''_{n-1} \arrow[r] & 0 \\
& \xi' \arrow[r, mapsto] & \partial\xi & &
\end{tikzcd}$$

gives

$$[\xi''] \longmapsto 0$$

$$\longrightarrow H_{n+1} \xrightarrow{j_*} H''_{n+1} \xrightarrow{\partial} H'_n \xrightarrow{i_*} H_n \xrightarrow{j_*} H''_n \xrightarrow{\partial} H'_{n-1} \xrightarrow{i_*} H_{n-1}$$

$\therefore\ \xi' = \partial\theta'$; Let $\theta' \longmapsto \theta$; $\xi - \theta \longmapsto \xi''$

$\cdot\ \partial\theta = \partial\xi \implies \partial(\xi - \theta] = 0$, i.e. $\xi - \theta$ is a cycle

Thus, $[\xi - \theta] \longmapsto [\xi'']$ $\square$

(4) Exactness at H_n'

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$$\begin{tikzcd}
 & & \tilde{\theta} & \tilde{\theta}'' & \\
0 \arrow[r] & C_{n+1}' \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d, "\xi"] & C_{n+1}'' \arrow[r] \arrow[d, "\xi''"] & 0 \\
0 \arrow[r] & C_n' \ \theta' \arrow[r] \arrow[d] & \theta \ C_n \arrow[r] \arrow[d] & C_n'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}' \arrow[r] & C_{n-1} \arrow[r] & C_{n-1}'' \arrow[r] & 0 \\
 & \xi' & \partial\xi & &
\end{tikzcd}$$

gives

$$\begin{tikzcd}
 & & & \in [\xi'] & & [\xi''] \arrow[r, mapsto] & 0 & \\
{} \arrow[r] & H_{n+1} \arrow[r, "j_*"] & H_{n+1}'' \arrow[r, "\partial"] & H_n' \arrow[r, "i_*"] & H_n \arrow[r, "j_*"] & H_n'' \arrow[r, "\partial"] & H_{n-1}' \arrow[r, "i_*"] & H_{n-1}
\end{tikzcd}$$

(a) $i_* \circ \partial = 0$; $[\xi''] \xmapsto{\partial} [\xi'] \xmapsto{i_*} [\partial\xi] = 0$ (shifted down in picture)

(b) $i_*([\theta']) = 0$; $\theta' \mapsto \theta$, then $\theta = \partial\tilde{\theta}$, $\tilde{\theta} \mapsto \tilde{\theta}''$; $\partial\tilde{\theta}'' = 0$

. By defn. of ∂, $\partial[\tilde{\theta}''] = [\theta']$ $\square$

Exact sequence in singular homology:

Let (X, A) be a pair of spaces.

Then we have a short exact sequence

$$0 \to C_*(A) \xrightarrow{i_\#} C_*(X) \xrightarrow{j_\#} C_*(X, A) \to 0$$

this gives a long exact sequence in homology

$$\cdots \to H_{n+1}(X,A) \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$$

Further, if $f : (X, A) \to (Y, B)$ is a map, then

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r] \arrow[d, "f_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial_*"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\arrow[r] & H_{n+1}(Y,B) \arrow[r] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y,B) \arrow[r, "\partial_*"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Cor: If f_* is an isomorphism on any two of $H_*(A)$, $H_*(X)$ and $H_*(X,A)$ (i.e., for all n), then it is also an isomorphism on the remaining one.

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r] \arrow[d, "f_*"] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\arrow[r] & H_{n+1}(Y,B) \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

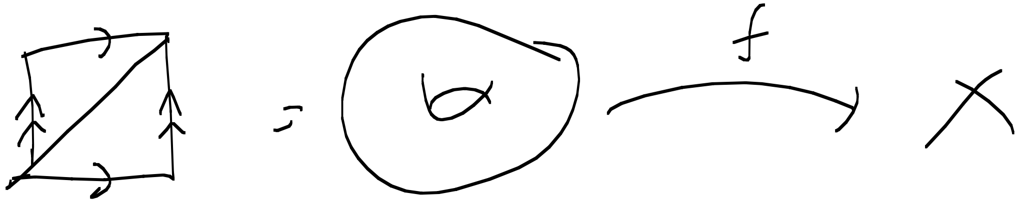
Pf: Use Five lemma, e.g.

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d, "\simeq"] & H_n(X) \arrow[r] \arrow[d, "\simeq"] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "\simeq"] & H_{n-1}(X) \arrow[d, "\simeq"] \\
H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y)
\end{tikzcd}$$

$\Rightarrow$ $H_n(X,A) \xrightarrow{\simeq} H_n(Y,B)$.

Geometric picture:

$H_2(X) \simeq$ Surfaces in X (oriented)

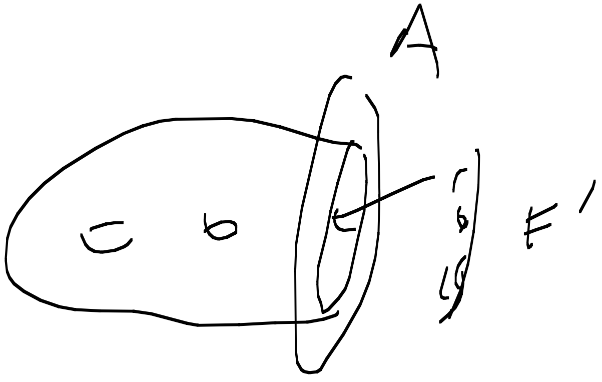


- Break surface into Δ^{les}
- Each Δ^{le} gives a singular simplex
- The sum of these is a cycle, as each edge is contained in two triangles which cancel

Conversely, we can glue together singular simplices of a cycle to get maps from surfaces.

Exactness: (X, A) ; $\begin{cases} H_2(X) = \text{closed surfaces } F \\ H_2(X,A) = \text{surfaces with boundary in } A, \text{ equal if they differ only within } A \end{cases}$

$(F, \partial F)$



$$\to H_2(A) \to H_2(X) \to H_2(X,A) \xrightarrow{\partial} H_1(A) \to H_1(X)$$

$$(F, \partial F) \mapsto \partial \qquad C \mapsto 0$$

Boundary map: $(F, \partial F) \to \partial F$

Fact: Any $\left(\begin{matrix}\text{curve} \\ \text{or 1-chain}\end{matrix}\right)$ in X which bounds a surface in X represents zero in homology & conversely.

Excision & Mayer-Vietoris (modulo a Lemma)

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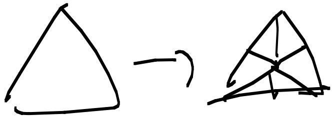
Small simplices lemma:

Let X be a topological space and $\mathcal{U}$ a collection of subsets of X whose interiors form an open cover.

Let $C_n^{\mathcal{U}}(X) = \{ \sum_{i=1}^{k} a_i \sigma_i : \sigma_i : \Delta^n \to X, a_i \in \mathbb{Z},$
$\forall i, \exists U_i \in \mathcal{U}$ s.t. $\sigma_i(\Delta^n) \subset U_i \}$

- $\partial_n^{\mathcal{U}} : C_n^{\mathcal{U}}(X) \longrightarrow C_{n-1}^{\mathcal{U}}(X)$ is the restriction of ∂.
- We have an inclusion $i : (C_*^{\mathcal{U}}, \partial_*^{\mathcal{U}}) \hookrightarrow (C_*, \partial_*)$

Lemma: $i_* : H_*^{\mathcal{U}}(X) \longrightarrow H_*(X)$ is an isomorphism.



Defn: A chain map inducing isomorphisms on homology is called a quasi-isomorphism.

Excision Axiom: · (X, A) − pair of spaces

- $B \subset A$ is such that $\bar{B} \subset \mathring{A}$

Thm: $i_* : H_*(X \setminus B, A \setminus B) \longrightarrow H_*(X, A)$ is an isomorphism.

Pf: Consider $\mathcal{U} = \{A, X \setminus B\}$, which satisfies the hypothesis of the lemma, as

$$\mathring{A} \cup \overset{\circ}{(X \setminus B)} = \mathring{A} \cup (X \setminus \bar{B}) = X.$$

Hence $H_*^{\mathcal{U}}(X) \xrightarrow{\cong} H_*(X)$.

$C_*(A) \hookrightarrow C_*^{\mathcal{U}}(X)$ and is a direct summand.

· Let $C_*^{\mathcal{U}}(X, A) = C_*^{\mathcal{U}}(X) / C_*(A)$. This is a free abelian group.

Lemma: $H_*^{\mathcal{U}}(X, A) \xrightarrow{\simeq} H_*(X, A)$

Pf: We have

$$\begin{tikzcd}
0 \arrow[r] & C_*(A) \arrow[r] \arrow[d] & C_*^{\mathcal{U}}(X) \arrow[r] \arrow[d] & C_*^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

which induces

$$\begin{tikzcd}
{} \arrow[r] & H_n(A) \arrow[r] \arrow[d, "\simeq"] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\simeq"] & H_n^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "\simeq"] & H_{n-1}^{\mathcal{U}}(X) \arrow[d, "\simeq"] \\
{} \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

By 5-lemma, $H_*^{\mathcal{U}}(X, A) \xrightarrow{\simeq} H_*(X, A)$.

Now consider $C_*^{\mathcal{U}}(X,A) = C_*^{\mathcal{U}}(X) / C_*^{\mathcal{U}}(A)$

- Any simplex in the basis of $C_*^{\mathcal{U}}(X)$ is either
  - Contained in $X \setminus B$

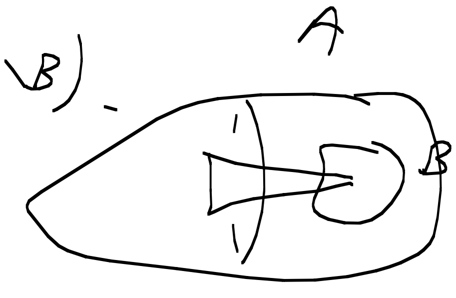
  or
  - Contained in $A. \Rightarrow 0$ in $C_*^{\mathcal{U}}(X,A)$.

$$\therefore \quad C_n^{\mathcal{U}}(X,A) = \frac{\{\Sigma a_i\sigma : \Delta^n \to X \setminus B : a_i \in \mathbb{Z}\}}{\{\Sigma a_i\sigma : \Delta^n \to A \setminus B : a_i \in \mathbb{Z}\}}$$

$$= \frac{C_n(X \setminus B)}{C_n(A \setminus B)} = C_n(X \setminus B, A \setminus B)$$

Thus

$$H_*(X,A) \cong H_*^{\mathcal{U}}(X,A) = H_*(X \setminus B, A \setminus B).$$



Chain Homotopy (Algebraisation of homotopy between maps)

Defn: Let $\varphi_*, \psi_* : (C_*, \partial_*) \longrightarrow (C'_*, \partial'_*)$ be chain homomorphisms between chain complexes.

A chain homotopy from φ to ψ is a collection of homomorphisms T_*,

$$T_n : C_n \longrightarrow C'_{n+1},$$

s.t. $$\partial'_{n+1} \circ T_n + T_{n-1} \circ \partial_n = \psi_n - \varphi_n$$

$$\begin{tikzcd}
C_{n+2} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d, "\varphi_n"', "\psi_n"] \arrow[dl, "T_n"'] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, "T_{n-1}"'] & {} \\
C'_{n+2} \arrow[r] & C'_{n+1} \arrow[r, "\partial'"'] & C'_n \arrow[r] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

$$\underbrace{\partial(D \times I)}_{\partial T} = \underbrace{(D \times \partial I)}_{\psi - \varphi} \cup \underbrace{(\partial D \times I)}_{T \circ \partial}$$

Theorem: If $\varphi_\#, \psi_\# : C_* \longrightarrow C'_*$ are chain homotopic chain homomorphisms, then the induced homomorphisms

$$\varphi_*, \psi_* : H_* \longrightarrow H'_*$$

are equal.

Pf: Let $[\xi] \in H_n$ and let T_* be the chain homotopy from φ to ψ.

Then
$$\psi_n(\xi) - \varphi_n(\xi) = \partial T(\xi) + T\partial(\xi)$$
$$= \partial T(\xi)$$

$$\therefore \quad [\psi_n(\xi)] - [\varphi_n(\xi)] = [\partial T(\xi)] = 0. \quad \square$$

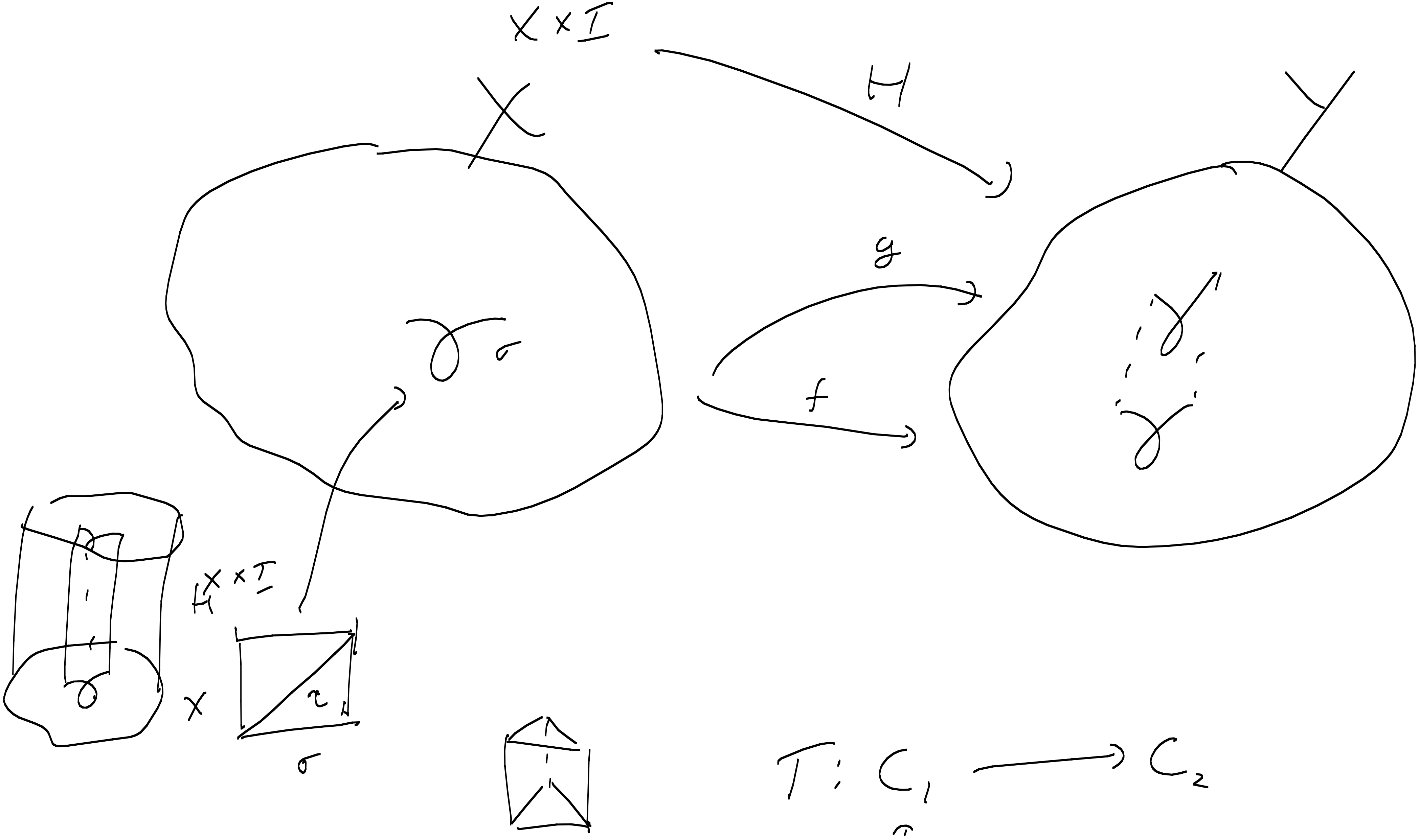
Defn: $\varphi_\#: C_* \longrightarrow C_*'$ is a chain homotopy equivalence if $\exists\ \psi_\# : C_*' \longrightarrow C_*$ s.t. $\varphi \circ \psi$ and $\psi \circ \varphi$ are chain homotopic to the identity.

Cor: If $\varphi_\# : C_* \longrightarrow C_*'$ is a chain homotopy equivalence, then the induced homomorphisms

$$\varphi_* : H_* \longrightarrow H_*'$$

are isomorphisms.

Lemma: If $f, g : X \longrightarrow Y$ are homotopic maps, then $f_\#, g_\# : C_*(X) \longrightarrow C_*(Y)$ are chain homotopic.



$$T : \underset{\tilde{\sigma}}{C_1} \longrightarrow C_2$$

$\sigma : \Delta^1 \longrightarrow X$

$\sigma \times I|_{\tau} : \Delta^1 \times I|_{\tau} \longrightarrow \cancel{X} \times I \xrightarrow{H} Y$

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Lemma: Let $f, g : X \to Y$ be maps and let $F : X\times I \to Y$ be a homotopy from f to g. Then $f_\#$ and $g_\#$ are chain homotopic, $f_\#, g_\# : C_*(X) \to C_*(Y)$.

Pf:

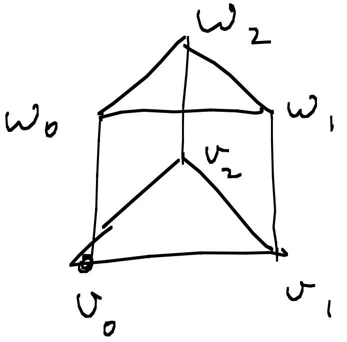
Given a homotopy $F:X\times I\to Y$ from f to g, we can define *prism operators* $P:C_n(X)\to C_{n+1}(Y)$ by

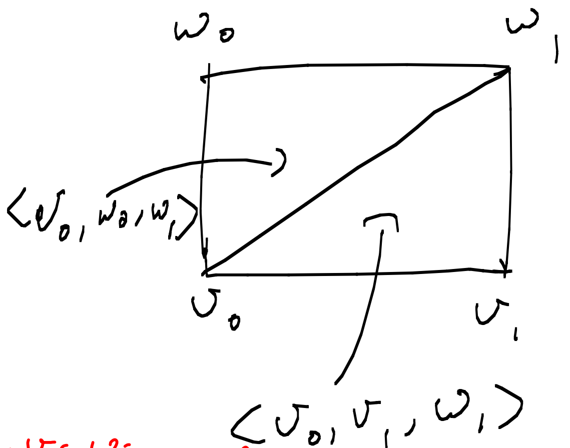
singular n-simplex

$$P(\sigma)=\sum_i(-1)^i F\circ\overbrace{(\sigma\times\mathbb{1})}^{\Delta^n\times I\to X\times I}|\underbrace{[v_0,\cdots,v_i,w_i,\cdots,w_n]}_{\to\ \text{subset of}\ \Delta^n\times I}$$

- We shall show $\partial \underbrace{P}_{C_n\to C_{n+1}} + P\partial = f_\# - g_\#$

Here,





- Let the vertices of Δ^n be $e_0, \dots, e_n$
- Let $v_i = e_i \times \{0\}$, $w_i = e_i \times \{1\}$

These are vertices of $\Delta^n \times [0,1]$.

Ex: $\Delta\times I = \bigcup_{i=0}^{n} \langle v_0, \dots, v_i, w_i, \dots, w_n\rangle$

Thus,

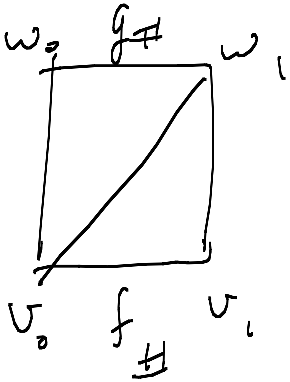
$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ \left[\overbrace{(\sigma \times \mathbb{1})\Big|\langle v_0, \dots, v_i, w_i, \dots, w_n \rangle}^{\Delta^{n+1} \longrightarrow X \times I} \right]$$

Lemma: $\partial P(\sigma) + P\partial(\sigma) = f_{\#}(\sigma) - g_{\#}(\sigma)$

Pf:
$$\partial P(\sigma) = \sum_{i=0}^{n} (-1)^i \left\{ \sum_{j=0}^{i-1} (-1)^j F \circ (\sigma \times \mathbb{1})\Big| \langle v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n \rangle \right\} \text{I}$$

$$\text{II} \left\{ \begin{array}{l} + (-1)^i \; F \circ (\sigma \times \mathbb{1})\Big| \langle v_0, \dots, v_{i-1}, w_i, \dots, w_n \rangle \\ + (-1)^{i+1} \; F \circ (\sigma \times \mathbb{1})\Big| \langle v_0, \dots, v_i, w_{i+1}, \dots, w_n \rangle \end{array} \right.$$

$$\text{III} \left\{ + \sum_{j=i+1}^{n} (-1)^{j+1} F \circ (\sigma \times \mathbb{1}) \Big| \langle v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n \rangle \right\}$$



Type II: $\sum_{i=0}^{n} F \circ (\sigma \times \mathbb{1})\Big|_{\langle v_0, \dots, v_{i-1}, w_i, \dots, w_n \rangle} - F \circ (\sigma \times \mathbb{1}) \Big| \langle v_0, \dots, v_i, w_{i+1}, \dots, w_n \rangle$

Cancels except for $\underbrace{F(\sigma \times \mathbb{1})\Big|\langle v_0, \dots, v_n \rangle - F(\sigma \times \mathbb{1})\Big|_{\langle w_0, \dots, w_n \rangle}}_{\text{check}}$

Thus, type II terms give $f_\#(\sigma) - g_\#(\sigma)$

Further,

$$P\partial(\sigma) = p\left(\sum_{j=0}^{n} (-1)^j \sigma|_{\langle e_0, \dots, \hat{e}_j, \dots, e_n\rangle}\right)$$

$$= \sum_{j=0}^{n} (-1)^j \begin{cases} \sum_{i=0}^{j-1} (-1)^i F\circ(\sigma \times \mathbb{I})|_{\langle v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n\rangle} \\ + \sum_{i=j+1}^{n} (-1)^{i-1} F\circ(\sigma\times\mathbb{I})|_{\langle v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n\rangle} \end{cases}$$

Ex: This cancels with type I & III terms in $\partial P + P\partial$. $\left[\begin{array}{l}\text{The term } F\circ(\sigma\times\mathbb{I})|_{\langle v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n\rangle} \\ \text{has signs } \begin{cases} \cdot\ (-1)^{i+j} \text{ here} \\ \cdot\ (-1)^{i+j+1} \text{ earlier}\end{cases}\end{array}\right]$

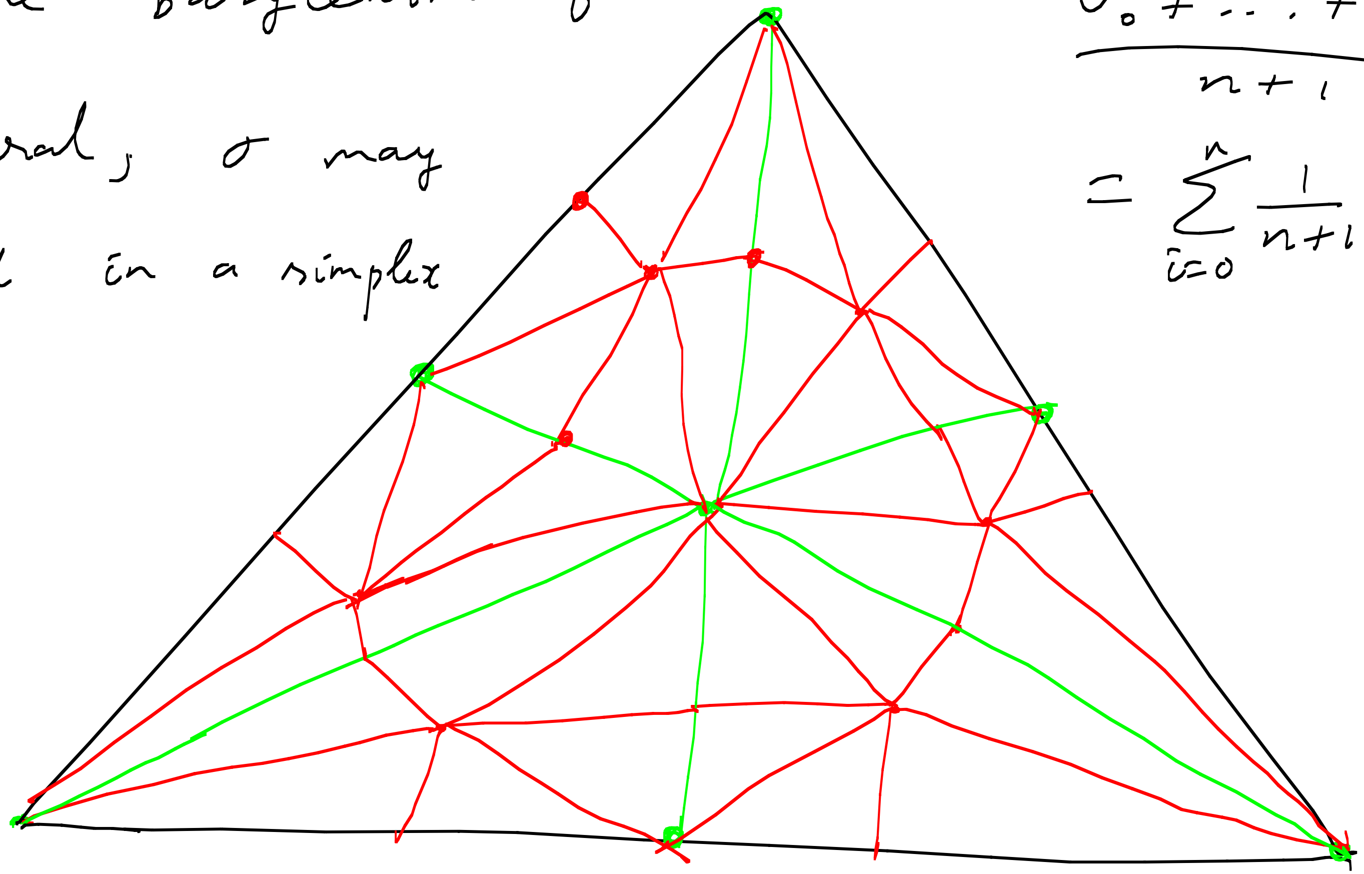
Thus, P gives the required chain homotopy.

Barycentric subdivision

Defn: Let $\sigma = \langle v_0, \ldots, v_n \rangle \subset \mathbb{R}^N$ be an n-simplex. Then the barycentre of σ is $\dfrac{v_0 + \ldots + v_n}{n+1}$.

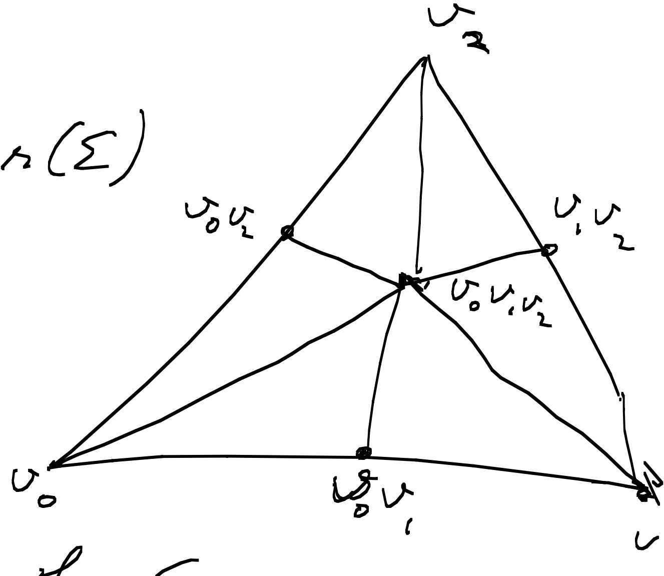
$$= \sum_{i=0}^{n} \frac{1}{n+1} v_i$$

- In general, σ may be contained in a simplex



Combinatorial description of barycentric subdivision.

- Let Σ be a simplicial complex
  - The simplices in Σ are partially ordered by inclusion.
  - The barycentric subdivision $br(\Sigma)$ is the poset complex of this partially ordered set.
    - Vertices : Simplices of Σ
    - Simplices : Chains of simplices of Σ.



Geometric map: $|br(\Sigma)| \longrightarrow |\Sigma|$ is given by

- $\underset{\langle v_0,\dots,v_k\rangle}{\overset{}{\sigma}} \in V(br(\Sigma)) \longmapsto$ barycentre of $\sigma = \sum_{i=0}^{k} \frac{1}{k+1} v_i$

$\sigma = \langle v_0,\dots,v_k\rangle$

- Extend linearly

Lemma: $C_*^{\mathcal{U}} \hookrightarrow C_*$ induces isomorphisms on homology

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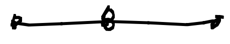
Barycentric Subdivision:

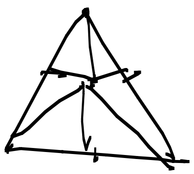
- Δ^n n-simplex
- We inductively define the barycentric subdivision
- For a point, $br(pt) = pt$.
- Given barycentric subdivision of faces of Δ^n,

Let $b = b_\Delta$ be the barycentre of Δ^n.

- Simplices of $br(\Delta^n)$: ($\partial\Delta^n$ = boundary of Δ^n)



  - $\langle w_0, \dots, w_k \rangle$, simplices of $br(\partial\Delta^n)$
  - $\langle b, w_0, \dots, w_k \rangle$, $\langle w_0, \dots, w_k \rangle$ simplex of $br(\partial\Delta^n)$

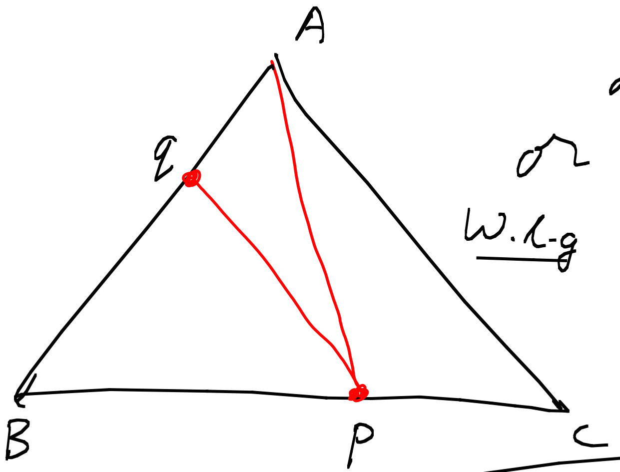


Lemma: If d = diameter of a simplex Δ^n, then the diameter of each simplex in $br(\Delta^n) \leq \frac{n}{n+1} d$

Decrease of diameter:

(1) Diameter = max distance between vertices

(by convexity of distance, i.e., of $f_p(q) = d(p,q)$)



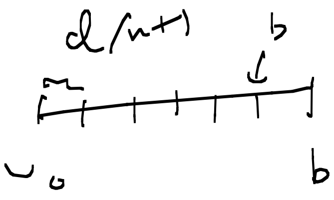
$d(p,q) \leq d(A,p)$

or $d(p,q) \leq d(B,p)$,

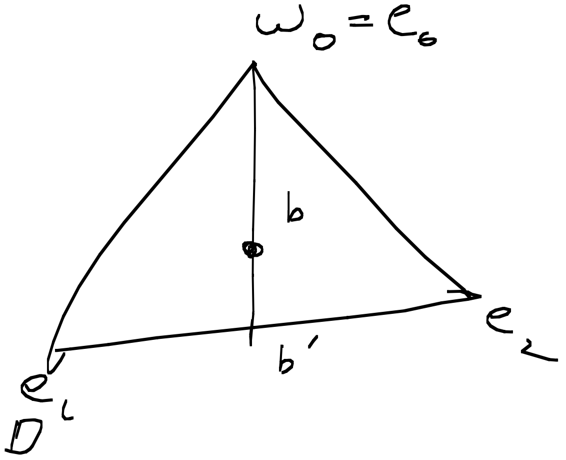
w.l.g $d(p,q) \leq d(A,p) \leq d(A,B)$

$$\boxed{\text{convexity: } f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)}$$

(2) Diameter in barycentric subdivision.



- For simplices $\langle w_0, .., w_k \rangle \subset \partial \Delta^n$, use induction
- For $\sigma = \langle b, w_0, .., w_k \rangle$, w.l.g. $\text{diam}(\sigma) = d(b, w_0)$, $w_0 =$
- Let b' be the barycentre of the face τ opposite w_0, i.e.



As $b = \frac{n}{n+1} \cdot b' + \frac{1}{n+1} w_0$,

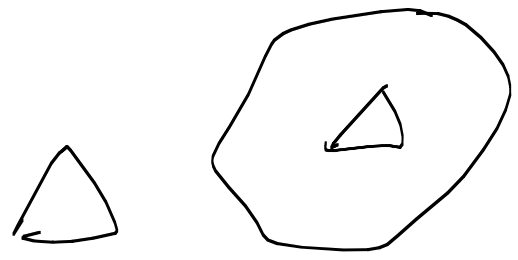
$d(w_0, b) = \frac{n}{n+1} d(w_0, b') \underset{n+1}{\leq n} \text{diam}(\Delta^n)$

Algebraic Operations:

Consider a convex set Z in $\mathbb{R}^N$.

Let $LC_k(Z)$ be the abelian group with basis linear simplices, i.e., $\sigma : \Delta^n \to Z$ linear.

- Let $LC_{-1}(Z) = \mathbb{Z}$ generated by the empty simplex $\emptyset$.

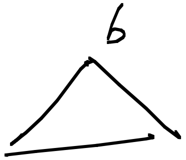


- We get a chain complex

<u>Cone</u>: Let $b \in Z$ be a point.

Define $b(\langle w_0 \ldots, w_k \rangle) = \langle b, w_0 \ldots, w_k \rangle$ as a homomorphism

$$b : LC_k \longrightarrow LC_{k+1}$$



Now $\partial b(\langle w_0, \ldots, w_k \rangle = \partial \langle b, w_0 \ldots, w_k \rangle$

$= \langle w_0, \ldots, w_k \rangle - b(\partial \langle w_0 \ldots, w_k \rangle)$

i.e., $\partial b + b \partial = \mathbb{1}$

Subdivision homomorphism : $S : LC_k \longrightarrow LC_k$.

- For a linear simplex $\sigma = \langle w_0 \ldots, w_k \rangle$, let b_σ be its barycentre

Define inductively: Define as identity on LC_{-1}

- Assume $S : LC_{k-1} \longrightarrow LC_{k-1}$ has been defined

Then define

$$S(\sigma) = b_\sigma(S(\underset{\substack{\uparrow \\ LC_{k-1}}}{\partial\sigma}))$$

Lemma: $\partial S = S\partial$

Pf: By induction on k

$$\begin{aligned} \partial S(\underset{\substack{\uparrow \\ LC_k}}{\sigma}) &= \partial b_\sigma(S(\partial\sigma)) \\ &= S\partial(\sigma) - b_\sigma(\partial \overbrace{S(\partial\sigma)}^{\in LC_{k-1}}) \end{aligned}$$

$\in LC_{k-1}$, so $\partial S(..) \overset{\shortparallel}{} S\partial(..)$

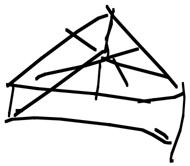
$$\partial S(\partial\sigma) \;\shortparallel\; b_\sigma(S\partial\partial\sigma) = 0 \quad \square$$

Thus, we have a chain homomorphism

$$S : LC_* \longrightarrow LC_*$$

Lemma: There is a chain homotopy T from S to the identity, i.e., $\partial T + T\partial = \mathbb{1} - S$

Pf: We define T inductively by



$$T = 0 \text{ for } n = -1$$

and $T\sigma = b_\sigma(\sigma - T\partial\sigma)$.

Then $\partial T\sigma + T\partial\sigma = \partial b_\sigma(\sigma - T\partial\sigma) + T\partial\sigma$

$$= \sigma - \cancel{T\partial\sigma} - b_\sigma\partial(\sigma - T\partial\sigma) + \cancel{T\partial\sigma}$$

$$= \sigma - b_\sigma(\partial\sigma) + b_\sigma\partial T(\partial\sigma)$$

Now $\partial T(\partial\sigma) + T\underset{\overset{\shortparallel}{0}}{\partial}(\partial\sigma) = \partial\sigma - S(\partial\sigma)$ [by induction]

Thus,

$$\partial T\sigma + T\partial\sigma = \sigma - b_\sigma(\partial\sigma) + b_\sigma \partial T(\partial\sigma)$$

$$= \sigma - b_\sigma(\partial\sigma) + b_\sigma(\partial\sigma) - b_\sigma(S(\partial\sigma))$$

$$= (\mathbb{1} - S)(\sigma)$$

$\square$

S and T on $C_*(X)$: $\begin{cases} \cdot\, S : C_n(X) \longrightarrow C_n(X) \\ \cdot\, T : C_n(X) \longrightarrow C_{n+1}(X) \end{cases}$

Let X be a topological space and $\sigma : \Delta^n \longrightarrow X$ a singular simplex.

- $\mathbb{1} : \Delta^n \longrightarrow \Delta^n \in LC_n(\Delta^n) \subset C_n(\Delta^n)$

Hence $S\mathbb{1} \in \underset{\underset{C_n(\Delta^n)}{\cap}}{LC_n(\Delta^n)}$ and $T\mathbb{1} \in \underset{\underset{C_{n+1}(\Delta^n)}{\cap}}{LC_{n+1}(\Delta^n)}$

Define $S(\sigma) = \sigma_\#(S\mathbb{1})$ and $T(\sigma) = \sigma_\#(T\mathbb{1})$.

<u>We have:</u> $S\partial = \partial S$, $\partial T + T\partial = \mathbb{1} - S$. (Exercise)

Iterated subdivision:

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- $S : C_*(X) \longrightarrow C_*(X)$ – chain homomorphism.
- $T : C_*(X) \longrightarrow C_{*+1}(X)$

- By iterating, $S^m(X) : C_*(X) \longrightarrow C_*(X)$ is a chain homomorphism
- A chain homotopy from $\mathbb{1}$ to S^m is given by

$$D_m = \sum_{i=0}^{m-1} T S^i$$

- $$\partial D_m + D_m \cdot \partial = \sum_{i=0}^{m-1} \partial T S^i + T S^i \partial = \sum_{i=0}^{m-1} \partial T S^i + T \partial S^i$$

$$= \sum_{i=0}^{m-1} (\mathbb{1} - S) S^i = \mathbb{1} - S^m$$

Shrinking diameter:

Let $\sigma : \Delta^n \longrightarrow X$ be a simplex

Then $\{\sigma^{-1}(\mathring{U}) : U \in \mathcal{U}\}$ is an open cover of Δ^n.

By Lesbesgue number theorem, $\exists\, \delta > 0$ s.t. if the diameter of a simplex $\tau \subset \sigma$ is at most δ, then

$$\tau \subset \sigma^{-1}(\mathring{U}) \text{ for some } U \in \mathcal{U} \Rightarrow \sigma(\tau) \subset \mathring{U},$$

i.e. $\sigma_{\#}(\tau) \subset C_n^{\mathcal{U}}(X)$

- For m large enough, and τ a simplex in $S^m(\mathbb{1})$,

$$\text{diam}(\tau) \leq \left(\frac{n}{n+1}\right)^m \cdot 1 < \delta$$

Hence, $S^m(\sigma) = \sigma_{\#}(S^m(\mathbb{1})) \subset C_n^{\mathcal{U}}(X)$ $\square$

Goal: Construct $\begin{cases} \cdot \text{ Chain homomorphism } C_*(X) \to C_*^{\mathcal{U}}(X) \\ \cdot \text{ Chain homotopy between this and the identity.} \end{cases}$

$$\begin{tikzcd} C_*^{\mathcal{U}} \arrow[r] & C_*(X) \arrow[l, bend left, dashed] \end{tikzcd}$$

Subtlety: The number of subdivisions needed for σ (i.e., m), depends on σ.

Hence, if $S := S^{m(\sigma)}(\sigma)$, $m(\sigma)$ minimum number of subdivisions needed for σ,

then S is <u>not</u> a chain homomorphism.

Strategy:

Let $D(\sigma) := D_{m(\sigma)}(\sigma)$ $\left[D_m = \sum_{i=0}^{m-1} TS^i \right]$

Then $$\underbrace{\partial D_{m(\sigma)}\cdot\sigma}_{\overset{\shortparallel}{\partial D\sigma}} + \underbrace{D_{m(\sigma)}\cdot\partial\sigma}_{\overset{\neq \text{ (in general)}}{D\partial\sigma}} = \mathbb{1} - S^{m(\sigma)}(\sigma)$$

· If τ is a simplex in $\partial\sigma$, $m(\tau) \leq m(\sigma)$.

$$\therefore \partial D\sigma + D\partial\sigma = \mathbb{1} - \overbrace{\left[\underbrace{S^{m(\sigma)}\sigma}_{S\sigma} + D_{m(\sigma)}\cdot\partial\sigma - D\partial\sigma\right]}^{\rho(\sigma)}$$

Let $\rho(\sigma) = S^{m(\sigma)}\cdot\sigma + D_{m(\sigma)}\cdot\partial\sigma - D\cdot\partial\sigma$

Lemma: $\rho(\sigma) \in C^{\mathcal{U}}(X)$

Lemma: $\rho(\sigma) = S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \in C_*^{\mathcal{U}}(X)$.

Pf: $S^{m(\sigma)}\sigma \in C_*^{\mathcal{U}}(X)$ by construction.

- If $\tau \subset \partial\sigma$ is a face of σ,

  $m(\tau) \leq m(\sigma)$, w.l.g. $m(\tau) < m(\sigma)$

Then $$D_{m(\sigma)}\tau - D\tau = \sum_{i=0}^{m(\sigma)-1} TS^i\tau - \sum_{i=0}^{m(\tau)-1} TS^i(\tau)$$
$$= \sum_{i=m(\tau)}^{m(\sigma)-1} TS^i(\tau)$$

- For $i \geqslant m(\tau)$, $S^i(\tau) \in C_n^{\mathcal{U}}(X)$, and $T: C_*^{\mathcal{U}}(X) \to C_{*+1}^{\mathcal{U}}(X)$

Hence $\rho(\sigma) \in C_n^{\mathcal{U}}(\sigma)$.

Thus, we have $\rho(\sigma) \in C_*^{\mathcal{U}}(X)$

and $\partial D(\sigma) + D\partial(\sigma) = \sigma - \rho(\sigma)$

$$\Rightarrow \partial D \partial(\sigma) = \partial\sigma - \partial\rho(\sigma)$$

$$\& \quad \partial D(\partial\sigma) + D\partial(\partial\sigma) = \partial\sigma - \rho(\partial\sigma)$$

$$\Rightarrow \partial D \partial \sigma = \partial\sigma - \rho(\partial\sigma)$$

$$\therefore \partial\rho(\sigma) = \rho(\partial\sigma)$$

Exercise: If $H : C_* \longrightarrow C_{*+1}$ is a collection of homomorphis
then $\varphi := \partial H + H\partial$ is a chain homomorphism

Thus, we have

- $i : C_*^{\mathcal{U}}(X) \longrightarrow C_*(X)$ – the inclusion, is a chain homomorphism.
- $\rho : C_*(X) \longrightarrow C_*^{\mathcal{U}}(X)$.
- $\partial D + D\partial = \mathbb{1} - \rho \Longrightarrow i \circ \rho : C_*(X) \longrightarrow C_*(X)$ satisfies $\partial D + D\partial = \underset{\text{on } C_*(X)}{\underline{\mathbb{1}}} - i \circ \rho$,

i.e. $i \circ \rho : C_*(X) \longrightarrow C_*(X)$ is chain homotopic to the identity.

- $\underline{\rho \circ i}$: If $\sigma \in C_*^{\mathcal{U}}(X)$, $\rho \circ i(\sigma) = \rho(\sigma)$

$$= S^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$$

$$= \sigma + 0 - 0 = \sigma, \quad \underline{\text{i.e.}}\ \rho \circ i = \mathbb{1}.$$

Thus, p is a <u>retraction</u>, so $i \circ p = \mathbb{1}$

& $p \circ i$ is homotopic to the identity.

Thus, $C^{\mathcal{U}}_*(X) \underset{p}{\overset{i}{\rightleftarrows}} C_*(X)$ are chain homotopy equiv. chain complexes.

Hence $i_* : H^{\mathcal{U}}_*(X) \xrightarrow{\cong} H_*(X)$.

This completes verification of Axioms for homology

Reduced Homology & Augmented chain complex.

- Augmented chain complex: Introduce $C_{-1} \cong \mathbb{Z}$

- $C_0(X) = \left\{ \sum_{i=1}^{k} a_i \cdot \sigma_i , \quad \sigma_i \quad 0\text{-simplices} \right\}$

Define $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ (Augmentation map).

$$\sum_{i=1}^{k} a_i \sigma_i \longmapsto \sum_{i=1}^{k} a_i$$

Lemma: $\cdots \to C_2 \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}$ is a chain complex.

- **Pf:** For $\sigma : [0,1] \to X$

$$\varepsilon \circ \partial_1(\sigma) = \varepsilon(\sigma(1) - \sigma(0)) = 1 - 1 = 0$$

Definition: The reduced homology $\tilde{H}_*(X)$ is the homology of the augmented chain complex.

- For $i \geq 1$, $\tilde{H}_i(X) = H_i(X)$

  For $i = 0$, $\tilde{H}_0(X) = \frac{\ker(\varepsilon)}{im(\partial_1)}$, $H_0(X) = \frac{C_0(X)}{im(\partial_1)}$

Now, $C_0(X) = \ker(\varepsilon) \oplus \mathbb{Z}$

[We have
$$\begin{tikzcd} 0 \arrow[r] & \ker(\varepsilon) \arrow[r] & C_0(X) \arrow[r, "\varepsilon"'] & \mathbb{Z} \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd}$$
which splits]

$\therefore$ $H_0(X) = \frac{\ker(\varepsilon) + \mathbb{Z}}{im(\partial_1)} = \tilde{H}_0(X) \oplus \mathbb{Z}$.

Some Algebra:

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Consider a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"] & A \arrow[r, "q"] \arrow[l, bend right, dashed, "r"'] & C \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0 \end{tikzcd}$$

of Abelian groups (or R-modules).

Theorem: The following are equivalent. (We say s.e.s splits)

(a) $A = B' \oplus C'$ with $i : B \xrightarrow{\simeq} B' \subset A$ and $q|_{C'} : \underset{\overset{\cap}{A}}{C'} \xrightarrow{\simeq} C$.

$\Uparrow$

$i(B) = B'$

(b) $\exists s : C \to A$ homomorphism s.t. $q \circ s : C \to C$ is $\mathbb{1}_C$

(c) $\exists r : A \to B$ s.t. $r \circ i : B \to B$ is $\mathbb{1}_B$

E.g.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times n"] & \mathbb{Z} \arrow[r] \arrow[l, bend right, dashed] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

does not split.

$(a) \Rightarrow (b)$ & (c) : Suppose $A = B' \oplus C'$,

$i : B \xrightarrow{\simeq} B' \subset A$, $q|_{C'} : C' \xrightarrow{\simeq} C$

Then, for

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"'] & A \arrow[r, "q"'] \arrow[l, bend right, dashed, "r"'] & C \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0 \end{tikzcd}$$

We define $s = (q|_{C'})^{-1} : C \longrightarrow C' \hookrightarrow A$

and $r : A \longrightarrow B$, $A = B' \oplus C'$

by $r_{|B'} = i^{-1}$

and $r_{|C'} \equiv 0$. (or any other homomorphism)

(b) ⇒ (c) :

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"'] & A \arrow[r, "q"'] & C \arrow[r] \arrow[l, bend right, "s"'] & 0 \end{tikzcd} \quad , \quad q \circ s = \mathbb{1}_C$$

We show $A = B' \oplus C'$, $\quad i : B \xrightarrow{\sim} B'$, $\quad q|_{C'} : C' \xrightarrow{\sim} C$

Let $B' = \operatorname{im}(i) = \ker(q)$.

$C' = \operatorname{im}(s)$.

Lemma: $A = B' \oplus C'$

Pf: Consider the homomorphism $B' \oplus C' \to A$ induced by inclusions.

Injectivity: Suppose $b' + c' \mapsto 0$, $b' \in B'$, $c' \in C'$,

i.e., $b' + c' = 0$ in A, i.e. $b' = -c'$.

$q(b') = 0 \Rightarrow q(c') = 0$

· As $c' \in C'$, $c' = s(c)$, $c \in C \Rightarrow c = q \circ s(c) = q(c') = 0$

$\Rightarrow c' = s(c) = 0 \Rightarrow b' = -c' = 0$.

<u>Surjectivity:</u> $B' = \mathrm{im}(i) = \ker(q)$, $C' = \mathrm{im}(s)$

$$\begin{tikzcd}
 & & a \in & & \\
0 \arrow[r] & B \arrow[r] & A \arrow[r, "q"'] & C \arrow[r] \arrow[l, bend right, "s"'] & 0 \\
 & & B' \oplus C' \arrow[u] & &
\end{tikzcd}$$

- Let $a \in A$, let $c = q(a) \in C$,

let $a' = s(q(a)) \in C'$

Then $q(a-a') = q(a) - \overset{\Pi_c}{\overset{\shortparallel}{q \circ s}} \circ q(a) = 0.$

$\therefore \quad a - a' = b' = i(b) \in B'$

Thus, $a = b' + c'$, $b' \in B'$, $c' \in C'$.

(c) <u>Retraction</u> $\Rightarrow$ ...

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"'] & A \arrow[r, "q"'] \arrow[l, bend right, dashed, "r"'] & C \arrow[r] & 0 \end{tikzcd}, \quad r \circ i = \mathbb{1}_B$$

Let $B' = i(B)$

$C' = \ker(r)$

Lemma: $B' \oplus C' \xrightarrow{\simeq} A$ (induced by inclusions)

Pf: Injectivity: Suppose $\overset{= i(b)}{b'} \in B'$, $c' \in C'$,

$b' + c' = 0$ (in A) $\Rightarrow b' = -c'$.

- As $c' \in C'$, $r(c') = 0 \Rightarrow r(b') = 0$.

$\Rightarrow r(i(b)) = 0 \Rightarrow b = 0 \Rightarrow b' = 0$

$\Rightarrow c' = 0$

Surjectivity:

$$\left.\begin{tikzcd}0 \arrow[r] & B \arrow[r, "i"'] & \underset{\overset{\uparrow}{a}}{A} \arrow[r, "q"'] \arrow[l, bend right, "r"'] & C \arrow[r] & 0\end{tikzcd}\right\}\begin{array}{l}B' = i(B)\\ C' = \ker(r)\end{array}$$

Let $a \in A$, consider $b = r(a)$,

let $a' = i(b) \in B'$

Then $r(a-a') = r(a) - r \circ i(b) = b - b = 0$

$\therefore \; a - a' \in \ker(r) = C'$

$\therefore \; a = \underset{B'}{\underset{\uparrow}{a'}} + \underset{C'}{\underset{\uparrow}{(a-a')}}$

$\square$

Thus, an s.e.s. may or may not split.

Thm: Suppose

$$0 \rightarrow B \xrightarrow[i]{} A \xrightarrow[q]{} F \rightarrow 0$$

is a s.e.s. of R-modules s.t. F is free. Then the s.e.s. splits.

Here,

Defn: An R-module F is said to be free with basis $x_1, \dots, x_n$ (or $\{x_\alpha\}_{\alpha \in A}$) if

Categorical defn: Given any function $\varphi : \{x_1, \dots, x_n\} \rightarrow M$, M an R-module, $\exists ! \, \Phi : F \rightarrow M$ R-module homomorphism s.t. $\Phi(x_i) = \varphi(x_i) \ \forall i$.

Explicit defn: Every $x \in F$ is uniquely $x = \sum_{i=1}^{n} a_i x_i$, $a_i \in R$

Pf of thm: We construct a splitting $s : F \to A$, $q \circ s = \mathbb{1}_K$

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"'] & A \arrow[r, "q"'] & F \arrow[r] \arrow[l, bend right, "s"'] & 0 \end{tikzcd}$$

Let $x_1, \dots, x_n$ be a basis,

we let $s(x_i) = y_i$ for some y_i s.t. $q(y_i) = x_i$

$q(y_i) = q \circ s(x_i) = x_i$

This defines a unique homomorphism $s : F \to A$.

Now, $q \circ s(x_i) = x_i = \mathbb{1}_F(x_i)$ $\forall x_i$ in basis

$\therefore$ $q \circ s = \mathbb{1}_F$.

E.g. $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$ are not free Abelian groups.

Qn: For what R-modules P is it true that every s.e.s.

$$0 \longrightarrow B \longrightarrow A \longrightarrow P \longrightarrow 0$$

splits.

Defn: P is said to be projective if the above holds

E.g. Free modules are projective.

Thm: P is projective iff $\exists$ an R-module Q s.t. $P \oplus Q = F$ is free.

E.g. $R = \mathbb{Z} \times \mathbb{Z}$, $P = \{(a,0) : a \in \mathbb{Z}\}$, $Q = \{(0,b) : b \in \mathbb{Z}\}$

$P \oplus Q = R$ which is a free R-module.

E.g. R is a free R-module with basis 1.

Pf of thm: Suppose $P \oplus Q = F$

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & A \arrow[r, "q"'] & P \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0 \end{tikzcd}$$ exact

As $q : A \twoheadrightarrow P$, $\tilde{q} = q \oplus \mathbb{1}_Q : A \oplus Q \twoheadrightarrow P \oplus Q = F$

Thus, there is a splitting $\tilde{s} : F \to A \oplus Q$
s.t. $\tilde{q} \circ \tilde{s} = \mathbb{1}_F$.

Claim: $\tilde{s}(P) \subset A$.

Pf: For $p \in P$, $\tilde{s}(p) = a \oplus x$, $a \in A$, $x \in Q$

$\Rightarrow p = \tilde{q} \circ \tilde{s}(p) = \tilde{q}(a \oplus x) = \underset{\overset{\cap}{P}}{q(a)} \oplus \underset{\overset{\cap}{Q}}{x} \Rightarrow x = 0$. ▫

Let $s : P \to A = \tilde{s}_{|P}$. Then $q \circ s = \mathbb{1}_P$.

Conversely, Construct $0 \to B \to F \to P \to 0$ - this splits

Reduced homology

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$$\longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

$$\tilde{H}_0(X) = \frac{\ker(\varepsilon)}{\mathrm{Im}(\partial_1)} \; ; \quad \tilde{H}_i(X) = H_i(X) \quad \forall i \geq 1$$

Hence
$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}_0(X) \arrow[r] & \overset{\cong C_0/\mathrm{Im}(\partial_1)}{H_0(X)} \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

- This splits:

Let $p \in X$ $(p \in C_0$, hence $[p] \in H_0)$

- Define $s(k) = k[p]^{\in H_0}$, then $\varepsilon \circ s = \mathbb{1}_{\mathbb{Z}}$.

Thus, $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$

Hence: $\tilde{H}_0(X) = \mathbb{Z}^{(\#\text{comps of } X) - 1}$. $\tilde{H}_0(pt) = 0.$

Thm: $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0 & \text{otherwise} \end{cases}$

- $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$,
- $S^0 = \{1, -1\} \Rightarrow \tilde{H}_k(S^0) = \begin{cases} \mathbb{Z}, & k = 0 \\ 0, & k \neq 0 \end{cases}$,
- $S^{-1} = \emptyset$

We proceed by induction using the lemma

Lemma: $\tilde{H}_{k+1}(S^{n+1}) = \tilde{H}_k(S^n)$, $k \geq 0$.

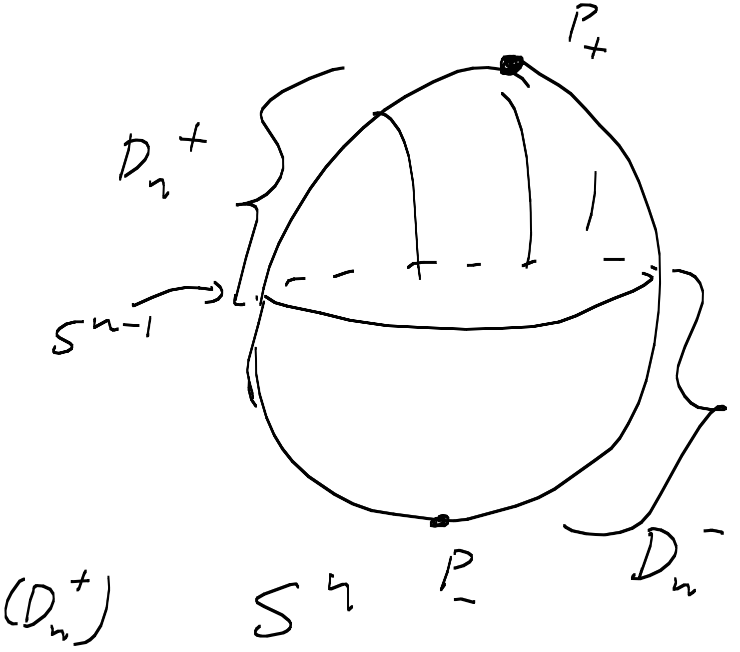
- $\tilde{H}_0(S^n) = 0$, $n \geq 1$.

Lemma: If $f:(X,A)\to(Y,B)$ s.t. $f:X\to Y$ and $f_{|A}:A\to B$ are homotopy equivalences, then $f_*:H_*(X,A)\xrightarrow{\simeq}H_*(Y,B)$.

Pf: Use
- homotopy axiom
- Exactness
- Five lemma

Rk: This applies when $(X,A)\subset(Y,B)$ and $X\subset Y$ and $A\subset B$ are deformation retracts, i.e.

$$H_*(X,A)\xrightarrow{i_*}H_*(Y,B).$$



Lemma: $\tilde{H}_{k+1}(D_n^+, S^{n-1}) \simeq \tilde{H}_k(S^{n-1})$

Pf: Use the long exact sequence in reduced homology

$$\begin{tikzcd}
\tilde{H}_{k+1}(D_n^+) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \tilde{H}_{k+1}(D_n^+, S^{n-1}) \arrow[r, "\partial"] & \tilde{H}_k(S^{n-1}) \arrow[r] & \tilde{H}_k(D_n^+) \arrow[d, phantom, "\shortparallel"] \\
0 & & & 0
\end{tikzcd}$$

Hence the homomorphism ∂ is an isomorphism.

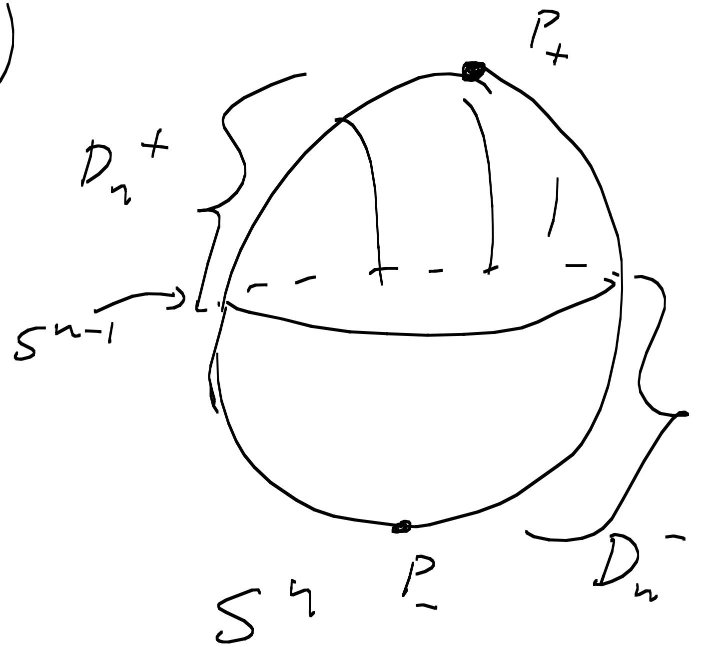
Lemma: $\tilde{H}_{k+1}(S^n, D_n^-) \simeq \tilde{H}_{k+1}(S^n)$. $\forall n$.

Pf: Use

$$\overset{\simeq 0}{\tilde{H}_{k+1}(D_n^-)} \longrightarrow \tilde{H}_{k+1}(S^n) \longrightarrow \tilde{H}_{k+1}(S^n, D_n^-) \longrightarrow \overset{= 0}{\tilde{H}_k(D_n^-)}$$

$\square$

<u>Lemma:</u> $\tilde{H}_k(S^n, D_n^-) \simeq \tilde{H}_k(D_n^+, S^{n-1})$



<u>Pf:</u> By excision,

$$\tilde{H}_k(S^n; D_n^-) \simeq \tilde{H}_k(S^n \setminus \{P_-\}, D_n^- \setminus \{P_-\})$$
$$\simeq \tilde{H}_k(D_n^+, S^{n-1})$$

as $(D_n^+, S^{n-1}) \hookrightarrow (S^n \setminus \{P_-\}, D_n^- \setminus \{P_-\})$

so that the inclusions $D_n^+ \subset S^n \setminus \{P_-\}$ and $S^{n-1} \subset D_n^- \setminus \{P_-\}$ are deformation retracts.

Thus,

$$\tilde{H}_k(S^{n-1}) \simeq \tilde{H}_{k+1}(D^n; S^{n-1}) \simeq \tilde{H}_{k+1}(S^n; D^n) \simeq \tilde{H}_{k+1}(S^n),$$

which is the main lemma.

Thm: $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0 & \text{otherwise} \end{cases}$

For $n \geq 1$,

$$H_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \text{ or } 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thm: (No retraction thm)

There is no map $r : D^n \longrightarrow S^{n-1}$ s.t. $r|_{S^{n-1}} = \mathbb{1}_{S^{n-1}}$

Pf: Suppose r exists, ie,

$$\begin{tikzcd} S^{n-1} \arrow[rr, "id"] \arrow[dr, hook] & & S^{n-1} \\ & D^n \arrow[ur, "r"'] & \end{tikzcd} \Rightarrow \begin{tikzcd} \overset{\mathbb{Z}}{\overset{\shortparallel}{\tilde{H}_{n-1}(S^{n-1})}} \arrow[rr, "id"] \arrow[dr, "i_*"'] & & \overset{\mathbb{Z}}{\overset{\shortparallel}{\tilde{H}_{n-1}(S^{n-1})}} \\ & 0 = H_{n-1}(D^n) \arrow[ur, "r_*"'] & \end{tikzcd},$$

which is impossible.

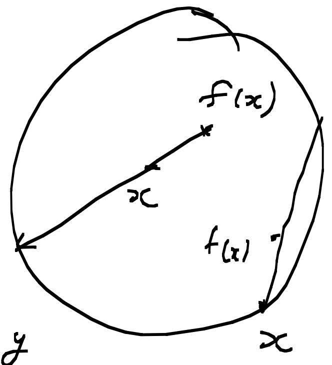
Brouwer fixed point theorem.

Any map $f: D^n \longrightarrow D^n$ has a fixed point.

Pf: Otherwise there is a retraction

$$r: D^n \longrightarrow S^{n-1}$$

given by $r(x) =$ first point after $f(x)$ where the ray from $f(x)$ through x touches the boundary,

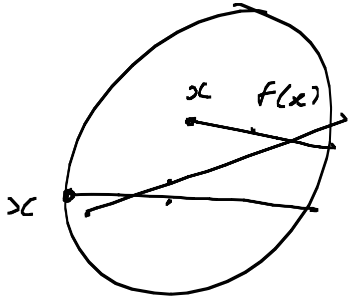


i.e. let $p_x(t) = t \cdot x + (1-t) f(x), \; t \geq 0$

let $t_0(x) = \inf \{ t > 0, \| p_x(t) \| = 1 \}$

and $r(x) = p_x(t_0(x))$.

Ex: This is continuous.



Invariance of dimension: $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^m$

$\Rightarrow n = m.$

Pf: $\mathbb{R}^n = \mathbb{R}^m \Rightarrow \mathbb{R}^n \setminus \{pt\} = \mathbb{R}^m \setminus \{pt\}$

$\Rightarrow S^{n-1} \overset{h.e.}{\sim} S^{m-1}$ (by taking defm. retracts)

$$\Rightarrow \underset{\mathbb{Z}}{\underset{\shortparallel}{\tilde{H}_{n-1}(S^{n-1})}} = \tilde{H}_{n-1}(S^{m-1})$$

$\Rightarrow n-1 = m-1 \Rightarrow n = m$

$\square$

Perron-Frobenius thm: Let A be an $n \times n$ matrix with all entries positive. Then A has a positive real eigenvalue with a corresponding eigenvector whose entries are positive.

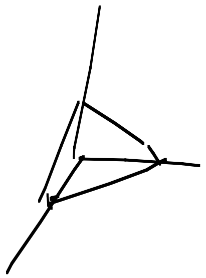
Pf: A induces a map on

$$\Delta^{n-1} = \{ (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \}$$

given by

$$f : \Delta^{n-1} \to \Delta^{n-1}$$

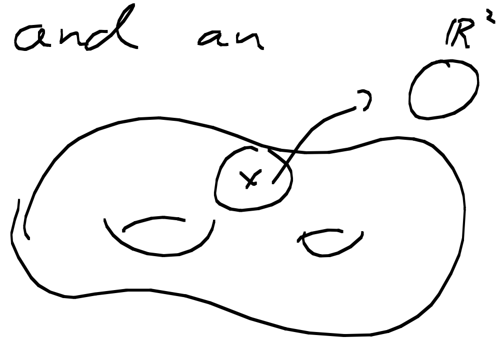
$$f(x_1, \dots, x_n) = \frac{(A_1(x_1, \dots, x_n), \dots, A_n(x_1, \dots, x_n))}{\sum A_i(x_1, \dots, x_n)}$$



where $A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_1(x_1, \dots, x_n) \\ \vdots \\ A_n(x_1, \dots, x_n) \end{pmatrix}$. Use Brower f.p.

Defn: A space X is an n-manifold if $\forall p \in X$, $\exists U \ni p$ open s.t. U is homeomorphic to $\mathbb{R}^n$.

Qn: Can $X \neq \emptyset$ be an n-manifold and an m-manifold for $m \neq n$.



Ans: No

Local homology: The local homology at $p \in X$ of X is $H_*(X, X \setminus \{p\})$.

Thm: If $U \subset X$ open, $p \in U$,

$$H_*(U, U \setminus \{p\}) = H_*(X, X \setminus \{p\})$$

Pf: Excise $X \setminus U \subset X \setminus \{p\}$

$\square$

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Cor: If X is an n-manifold, $n \geq 1$,

then $$H_k(X, X \setminus \{p\}) = \begin{cases} \mathbb{Z}, & k=n \\ 0 & \text{otherwise} \end{cases}$$

Pf: $$H_k(X, X\setminus\{p\}) \cong \underset{\substack{\| \\ H_k(D^n, D^n \setminus \{0\})}}{H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})} \cong \tilde{H}_{k-1}(S^{n-1})$$ $\square$

Cor: If $X \neq \emptyset$ is an n-manifold and an m-manifold,

then $n=m$.

Rk: A generator $\mu_x \in H_n(X, X\setminus\{p\})$ is called

an **orientation** for X at x.

Mayer-Vietoris Exact Sequence:

Let $X = V_1 \cup V_2$ with $V_i \subset X$ open. Then there is a long exact sequence in homology (which is natural)

$$\cdots \to H_{n+1}(X) \xrightarrow{\partial} H_n(V_1 \cap V_2) \longrightarrow H_n(V_1) \oplus H_n(V_2) \longrightarrow \underset{\substack{\wr\| \\ H_n^{\mathcal{U}}(X)}}{H_n(X)} \xrightarrow{\partial} H_{n-1}(V_1 \cap V_2) \to$$

Pf: Consider the s.e.s. of chain complexes

$$0 \longrightarrow C_*(V_1 \cap V_2) \xrightarrow{(i_1, -i_2)} C_*(V_1) \oplus C_*(V_2) \xrightarrow{j_1 + j_2} C_*^{\mathcal{U}}(X) \longrightarrow 0$$

where $\mathcal{U} = \{V_1, V_2\}$. (Here $C_*^{\mathcal{U}}$ = chains of small simplices)

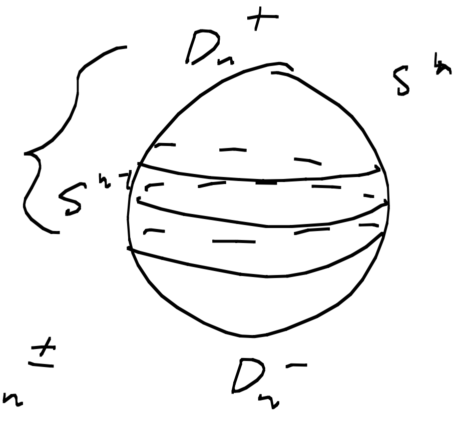
Exercise: This is exact.

□

E.g. We prove $H_{k+1}(S^n) = H_k(S^{n-1})$, $k \geq 1$

Using the Mayer-Vietoris sequence for

$V^\pm = \text{nbd}(D_n^\pm)$, $V^+ \cap V^- = \text{nbd}(S^{n-1})$,



observe that
- $V^\pm$ defm retracts to $D_n^\pm$
- $V^+ \cap V^-$ defm retracts to S^{n-1},

we have

$$\begin{tikzcd}
{} \arrow[r] & H_{k+1}(V_+ \cap V_-) \arrow[r] & H_{k+1}(V_+) \oplus H_{k+1}(V_-) \arrow[r] \arrow[d, phantom, "\parallel"] & H_{k+1}(S^n) \arrow[r, "\partial"] & H_k(V_+ \cap V_-) \arrow[r] \arrow[d, phantom, "\parallel"] & H_k(V_+) \oplus H_k(V_-) \arrow[d, phantom, "\parallel"] \\
 & & 0 & & H_k(S^{n-1}) & 0
\end{tikzcd}$$

Thus, $H_{k+1}(S^n) = H_k(S^{n-1})$

$\square$

Generator of $H_n(S^n)$ (from $H_{n-1}(S^{n-1})$)

$\tilde{H}_0(S^0) \xleftarrow{\partial} H_1(S^1) \xleftarrow{\partial} H_2(S^2)\Big)$ $\Big)$ $\underset{\overset{\wr\wr}{0}}{H_1(V_+)\oplus H_1(V_-)} \to H_1(S^1) \xrightarrow{\partial} \tilde{H}_0(S_0)$

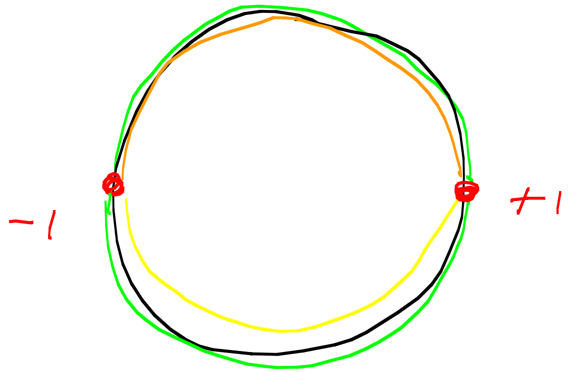
$\downarrow$

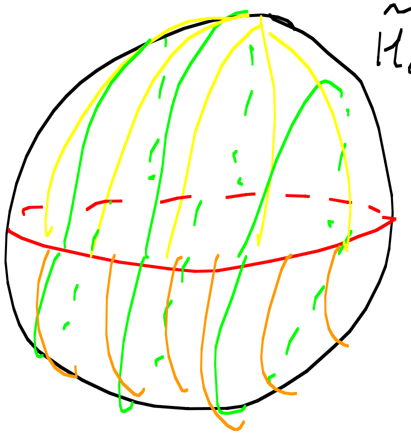
$\underset{\overset{\wr\wr}{0}}{\tilde{H}_0(V_+)\oplus \dots}$

$\wr$

generated by

$[1]-[-1] \in C_0$ ⟶





- We have a generator $\xi \in H_{n-1}(S^{n-1})$
- This is trivial in $H_{n-1}(D_n^{\pm})$. Hence $\xi = \partial \zeta_{\pm}$, $\zeta_{\pm} \in C_n(V_{\pm})$

the cycle
$\swarrow$
$\partial(\zeta_+ - \zeta_-) = 0$

$H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(V_+) \oplus H_{n-1}(V_-)$

Thm: Let $f: S^{n-1} \longrightarrow S^n$ be an injective map.
Then $S^n \setminus f(S^{n-1})$ has two components

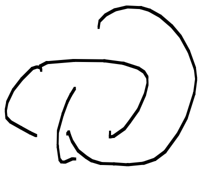
(Jordan-Brouwer Separation theorem)

In terms of homology

$$\tilde{H}_0(S^n \setminus f(S^{n-1})) = \mathbb{Z}$$

Lefschetz/Alexander duality

Thm: Suppose $f: S^k \to S^n$ is an injective map, $k \leq n-1$. Then



$$\tilde{H}_j(S^n \setminus f(S^{n-1})) = \begin{cases} \mathbb{Z} & \text{if } j = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$

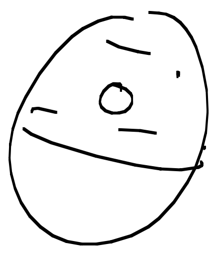
Main technical lemma:

Let $f: [0,1]^k \longrightarrow S^n$ be an injective map, $k \leq n$.

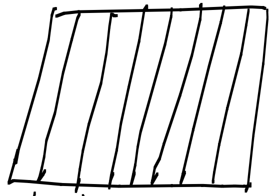
Then $\tilde{H}_j(S^n \setminus f([0,1]^k)) = 0 \quad \forall j$.

Pf: $\cdot$ For $k=0$, $S^n \setminus f([0,1]^k) = \mathbb{R}^n$, so the result holds.

$\cdot$ We proceed by induction.



Suppose $\xi \in C_j(S^n \setminus f([0,1]^k))$ is a cycle with $[\xi] \neq 0$ in $\tilde{H}_j(S^n \setminus f([0,1]^k))$.



By induction hypothesis, for $t \in [0,1]$,

$$[\xi] = 0 \text{ in } \tilde{H}_j(S^n \setminus f([0,1]^{k-1} \times \{t\})) = 0$$

$$\Rightarrow \xi = \partial \zeta_t, \quad \zeta_t \in C_{j+1}(S^n \setminus f([0,1]^{k-1} \times \{t\}))$$

Thus,

$$z = \partial \zeta_t \ , \ \zeta_t \in C_{j+1}(S^n \setminus f([0,1]^{k-1} \times \{t\})),$$

i.e., ζ_t is a $(j+1)$-chain in S^n disjoint from

$$f([0,1]^{k-1} \times \{t\}).$$

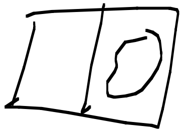
Ex

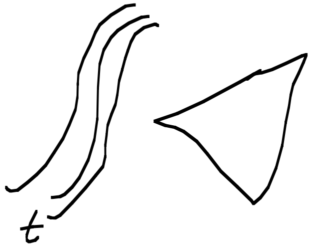
- As the image of each singular simplex is compact and ζ_t is a finite linear combination of such simplices

$\exists\, U_t \subset [0,1]$, $t \in U_t$ s.t.

$$\zeta_t \in C_{j+1}(S^n \setminus f([0,1]^{k-1} \times U_t)).$$

[<u>Formalise</u>: For a singular simplex with image Δ, consider $f^{-1}(\Delta)$.





Conclusion:

We have open sets $U_t \subset [0,1]$,

- chains ζ_t in S^n disjoint from

$$f([0,1]^{k-1} \times U_t)$$

s.t. $\xi = \partial \zeta_t$.

- $t \in U_t \Rightarrow U_t$ form an open cover of $[0,1]$,

By Lesbesgue number theorem,

$\exists\, \varepsilon > 0$ s.t. if $J \subset [0,1]$ has length $< \varepsilon$,

then $J \in U_t$ for some t

$$\Rightarrow \underset{\substack{\| \\ \partial \zeta_t}}{[\xi]} = 0 \quad \text{in} \quad H_j(S^n \setminus f([0,1]^{k-1} \times J))$$

To get a contradiction:

Step 1: We have assumed $[\xi] \neq 0$ in $H_n(S^n \setminus f([0,1]^k))$

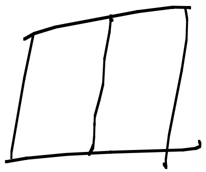
Let $I_1 = [0, 1/2]$, $I_2 = (1/2, 1]$.

Then $\underset{\substack{\parallel \\ V_j}}{S^n \setminus f([0,1]^{k-1} \times I_j)}$ is open for $j = 1, 2$,

and

$$V_1 \cap V_2 = S^n \setminus f([0,1]^k)$$

$$V_1 \cup V_2 = S^n \setminus f([0,1]^{k-1} \times \{\tfrac{1}{2}\})$$



So Mayer-Vietoris gives

$$\underset{\substack{\wr\wr \\ 0 \text{ by} \\ \text{induction hypothesis}}}{\tilde{H}_{j+1}(V_1 \cup V_2)} \to \underset{\substack{\ni \\ [\xi] \neq 0}}{\tilde{H}_j(V_1 \cap V_2)} \to \tilde{H}_j(V_1) \oplus \tilde{H}_j(V_2) \qquad \partial.$$

$\Rightarrow [\xi] \neq 0$ in one of V_1 & V_2

Iterating: Assume $[\xi] \neq 0$ in $\tilde{H}_j(\underset{\overset{\shortparallel}{V_1}}{S^n \setminus f([0,1]^{k-1} \times I_1)})$

We subdivide I_1 into I_{11} & I_{12} and repeat the argument.

Hence: There is an interval J_2 of length $1/4$ s.t. $[\xi] \neq 0$ in $\tilde{H}_j(S^n \setminus f([0,1]^{k-1} \times J_2))$

Repeating, $\forall p$ there is an interval J_p of length 2^{-p} s.t. $[\xi] \neq 0$ in $\tilde{H}_j(S^n \setminus f([0,1]^{k-1} \times J_p))$.

This gives a contradiction.

$\square$

Main technical lemma via Compact Support

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Axiom of compact support

Suppose X is a topological space.

(a) If $\alpha \in H_n(X)$ is a homology class, then $\exists\ K \subset X$ compact and $\alpha_K \in H_n(K)$ s.t.

$$\alpha = i_*(\alpha_K), \quad i : K \hookrightarrow X \text{ inclusion.}$$

(b) Further, if α as above is zero in homology, then there is a compact set $L \supset K$ s.t.

$$j_*(\alpha_K) = 0, \quad j : K \hookrightarrow L \text{ inclusion map.}$$

Pf: Choose represantitive cycles and boundaries. $\square$

Pf of inductive step:

To show: $f: [0,1]^k \longrightarrow S^n$, then $\underset{\substack{\shortparallel \\ 0}}{\tilde{H}_*(S^n \setminus f([0,1]^k))}$,

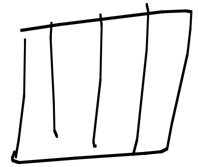
assuming this for $k-1$.

Hence, $\forall c \in [0,1]$, $\tilde{H}_*(S^n \setminus f([0,1]^{k-1} \times \{c\})) = 0$

Suppose $\alpha \in \tilde{H}_k(S^n \setminus f([0,1]^k))$, then by Axiom of compact support, $\alpha \in \tilde{H}_k(K)$, $K \subset S^n \setminus f([0,1]^k)$ cpt.

In particular, $\forall c \in [0,1]$, $K \cap f([0,1]^{k-1} \times \{c\}) = \emptyset$,

i.e. $f^{-1}(K) \cap (\underset{\underset{[0,1]^k}{\cap}}{[0,1]^{k-1}} \times \{c\}) = \emptyset$.



Now, $\alpha = 0$ in $\tilde{H}_k(S^n \setminus f([0,1]^{k-1} \times \{c\})) \Rightarrow \exists L_c \supset K$ cpt, $L_c \subset S^n \setminus f([0,1]^{k-1} \times \{c\})$ s.t. $\alpha = 0$ in $\tilde{H}_k(L_c)$.

$L_c \subset S^n \setminus f([0,1]^{k-1} \times \{c\})$, i.e. $f^{-1}(L_c) \cap [0,1]^{k-1} \times \{c\} = \emptyset$.

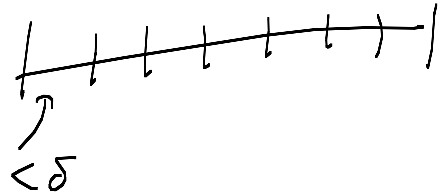
__Hence,__ for some open nbd. I_c of $\{c\}$,

$$L_c \subset S^n \setminus f([0,1]^{k-1} \times I_c).$$

. The sets I_c form an open cover of $[0,1]$.

Hence $\exists\, \delta > 0$ s.t. every interval J of length $\leq \delta$ is contained in some I_c.

$$(*_\delta)\begin{cases} \therefore\ \alpha = 0 \text{ in } H_k(S^n \setminus f([0,1]^{k-1} \times J)) \\ \text{if } |J| \leq \delta \end{cases}$$

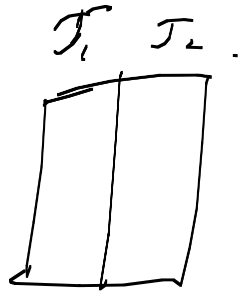


$<\delta$

__Lemma:__ $(*)_\delta \Rightarrow (*)_{2\delta}$

Lemma: $(*)_\delta \Rightarrow (*)_{2\delta}$

Pf: Suppose $\text{length}(J) \leq 2\delta$, then $J = J_1 \cup J_2$, $\text{length}(J_i) \leq \delta$, $J_1 \cap J_2 = \{c\}$.



Let $V_i = S^n \setminus f([0,1]^{k-1} \times J_i)$

Then $V_1 \cap V_2 = S^n \setminus f([0,1]^{k-1} \times J)$

& $V_1 \cup V_2 = S^n \setminus f([0,1]^{k-1} \times \{c\})$

The Mayer-Vietoris sequence gives

$$\begin{tikzcd}
\tilde{H}_{k+1}(V_1 \cup V_2) \arrow[r] \arrow[d, equal] & \tilde{H}_k(V_1 \cap V_2) \arrow[r] & \tilde{H}_k(V_1) \oplus \tilde{H}_k(V_2) \arrow[d, equal] \\
0 & \alpha \arrow[u, phantom, sloped, "\in"] & 0 \quad 0 \\
 & & \text{by } (*)_\delta
\end{tikzcd}$$

$\Rightarrow \alpha = 0$ in $\tilde{H}_k(S^n \setminus f([0,1]^{k-1} \times J))$.

$\square$.

Alexander duality: $f: S^k \to S^n$ embedding, $k<n$,

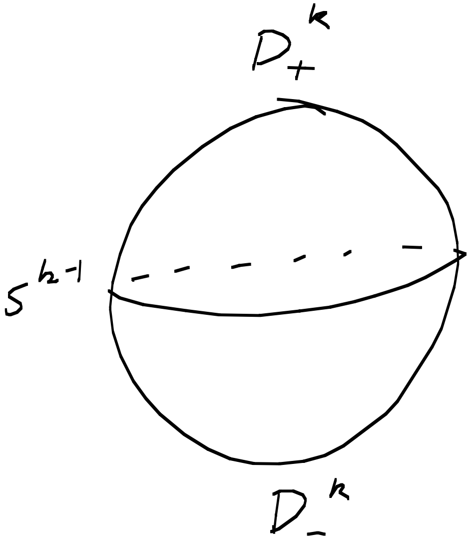
$$\tilde{H}_m(S^n \setminus f(S^k)) = \begin{cases} \mathbb{Z}, & m = n-k-1 \quad (k = n-m-1) \\ 0 & \text{otherwise} \end{cases}$$

Pf: Induction on k.

$k=0$: $S^n \setminus f(S^0) = \mathbb{R}^n \setminus \{0\} \overset{h.e.}{\sim} S^{n-1}$

Suppose we know this for S^{k-1}.

Let D^k_+ & D^k_- be the closed hemispheres in S^k; $S^{k-1} = D_+^k \cap D_-^k$



By Main lemma: $\tilde{H}_*(S^n \setminus f(D^k_\pm)) = 0$.

[We say $S^n \setminus f(D^k)$ is acyclic)

Let $V_\pm = S^n \setminus f(D^k_\pm)$.

$$V_+ \cap V_- = S^n \setminus f(S^k)$$

$$V_+ \cup V_- = S^n \setminus f(S^{k-1}).$$

$\therefore$ By Mayer - Vietoris

$$\underset{\underset{0}{\shortparallel}}{\tilde{H}_{m+1}(-)\oplus\cdots} \longrightarrow \tilde{H}_{m+1}(V_+ \cup V_-) \longrightarrow \tilde{H}_m(V_+ \cap V_-) \longrightarrow \underset{\underset{0}{\shortparallel}}{\tilde{H}_m(V_+) \oplus \tilde{H}_m(V_-)}$$

$$\therefore \quad H_m(S^n \setminus f(S^k)) = H_{m+1}(S^n \setminus f(S^{k-1})).$$

$$= \begin{cases} \mathbb{Z}, & m+1 = n-(k-1)-1 \Leftrightarrow m = n-k-1. \\ 0 \end{cases}$$

Rk: $\tilde{H}_m(S^n \setminus f(S^k)) = \tilde{H}_{n-m-1}(S^k)$.

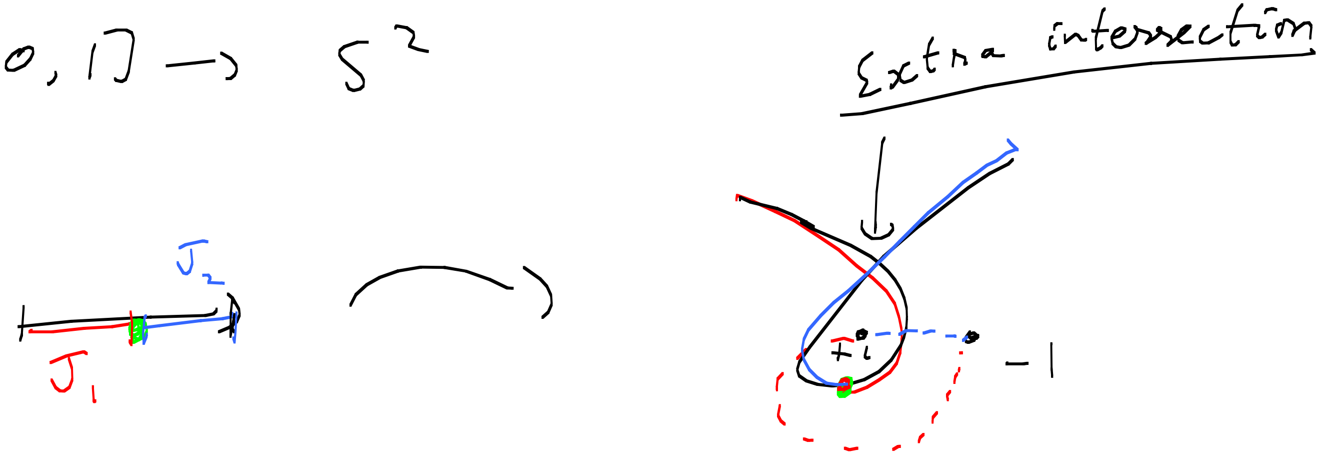
☺

Ex: Look for Alexander horned sphere on the web.

E.g. f not injective

$$f : [0,1] \to S^2$$

$f :$



Step: From J_1 & J_2 to $J_1 \cup J_2$

$S^2 \setminus f(J_1)$ & $S^2 \setminus f(J_2)$ are contractible

$$\tilde{H}_0(S^2 \setminus f(J_1 \cup J_2)) = \mathbb{Z}$$

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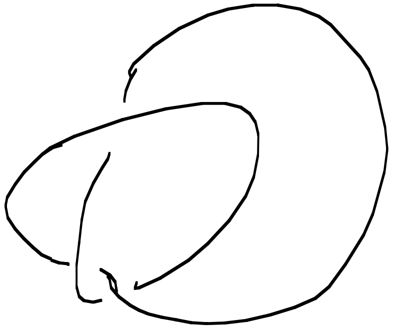
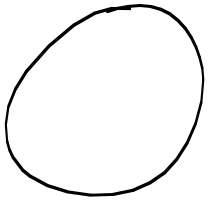
Examples:





- A knot $\overset{K}{\wedge}$ is an embedding $f : S^1 \longrightarrow S^3$.

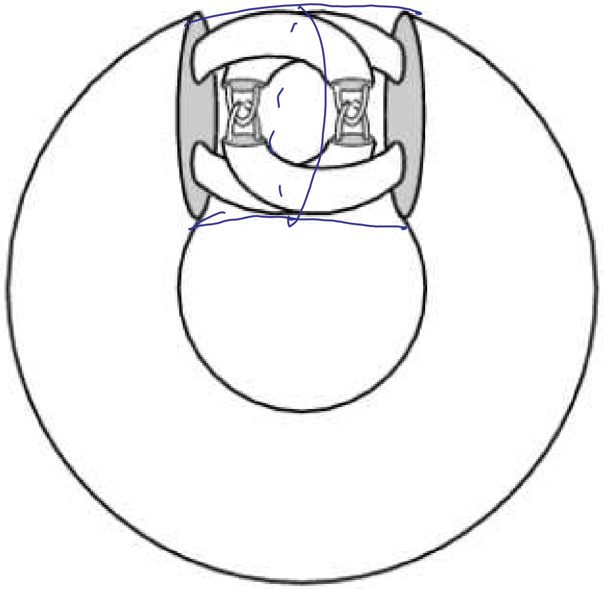
Cor: $H_k(S^3 \setminus f(S^1)) = \begin{cases} \mathbb{Z}, & k = 1, 0 \\ 0 & \text{otherwise.} \end{cases}$

Note: For two knots K_1 & K_2, (S^3, K_1) & (S^3, K_2) are not homeomorphic as pairs.

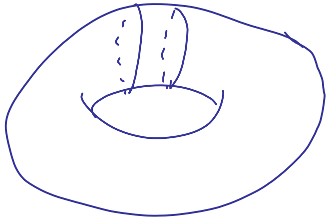
In fact, in general $\pi_1(S^3 \setminus K_1) \neq \pi_1(S^3 \setminus K_2)$.

Alexander horned Sphere.

- This is an embedding of S^2 in S^3, $f: S^2 \to S^3$.
- We know $S^3 \setminus f(S^2)$ has two components.
- One of these is a ball (and its closure is a closed ball)
- The other is not a ball.

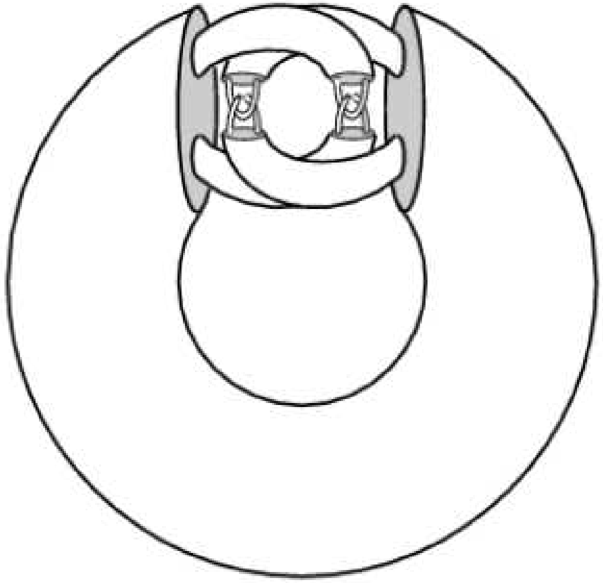


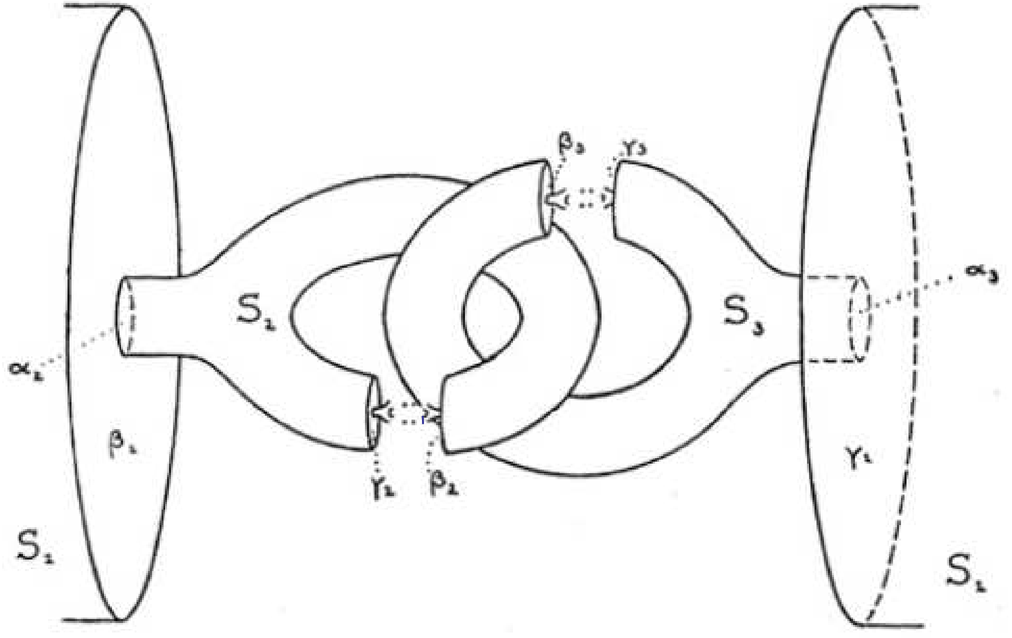
Step 1: Consider a standard solid torus



Step 2: . Delete a meridinal disc – gives a sphere,

. Introduce linked punctured tori.



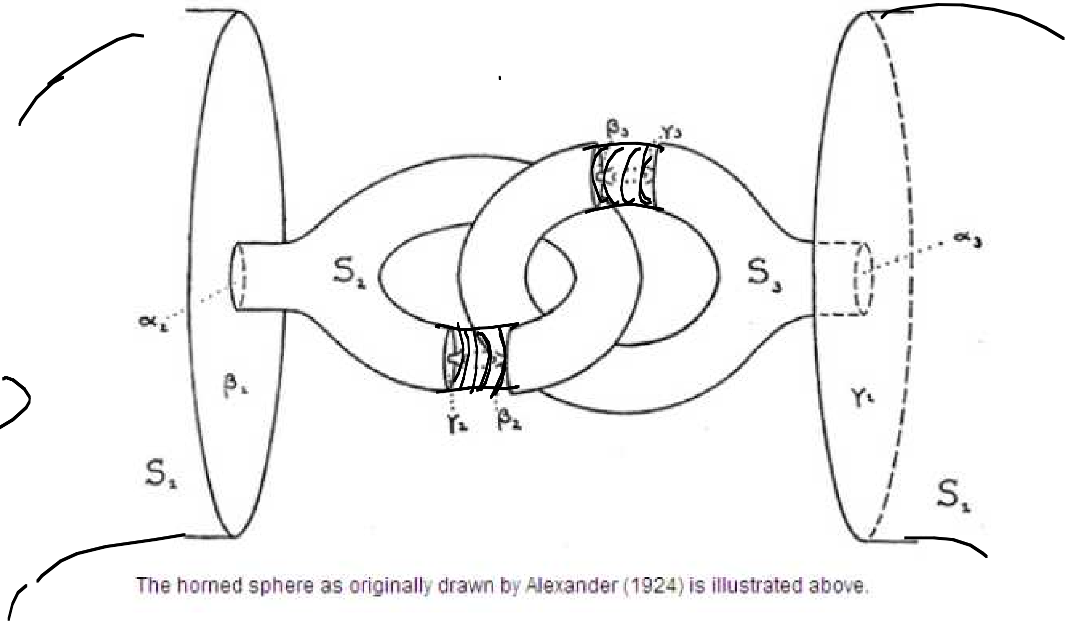


The horned sphere as originally drawn by Alexander (1924) is illustrated above.

Step 1: Take a solid torus

Step 2: { · Delete Meridinal disc
· Glue in solid tori

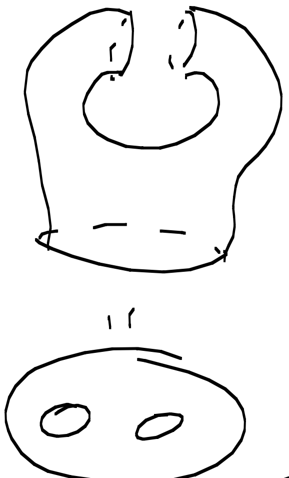
· Iterate this and consider the intersection

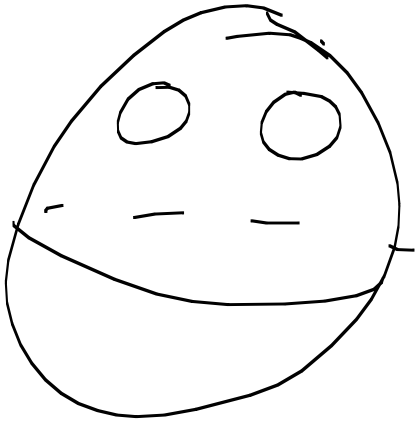


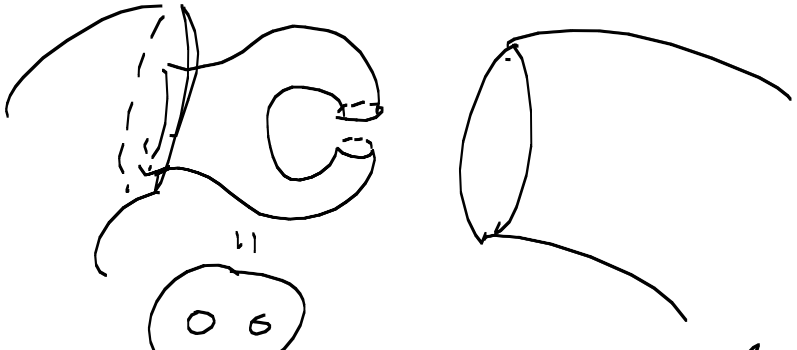
The horned sphere as originally drawn by Alexander (1924) is illustrated above.

Lemma: The outer boundary is the image of a sphere.

Pf: The outer boundary of solid torus with meridonal discs removed is the image of $S^2 \setminus 2$ open disks

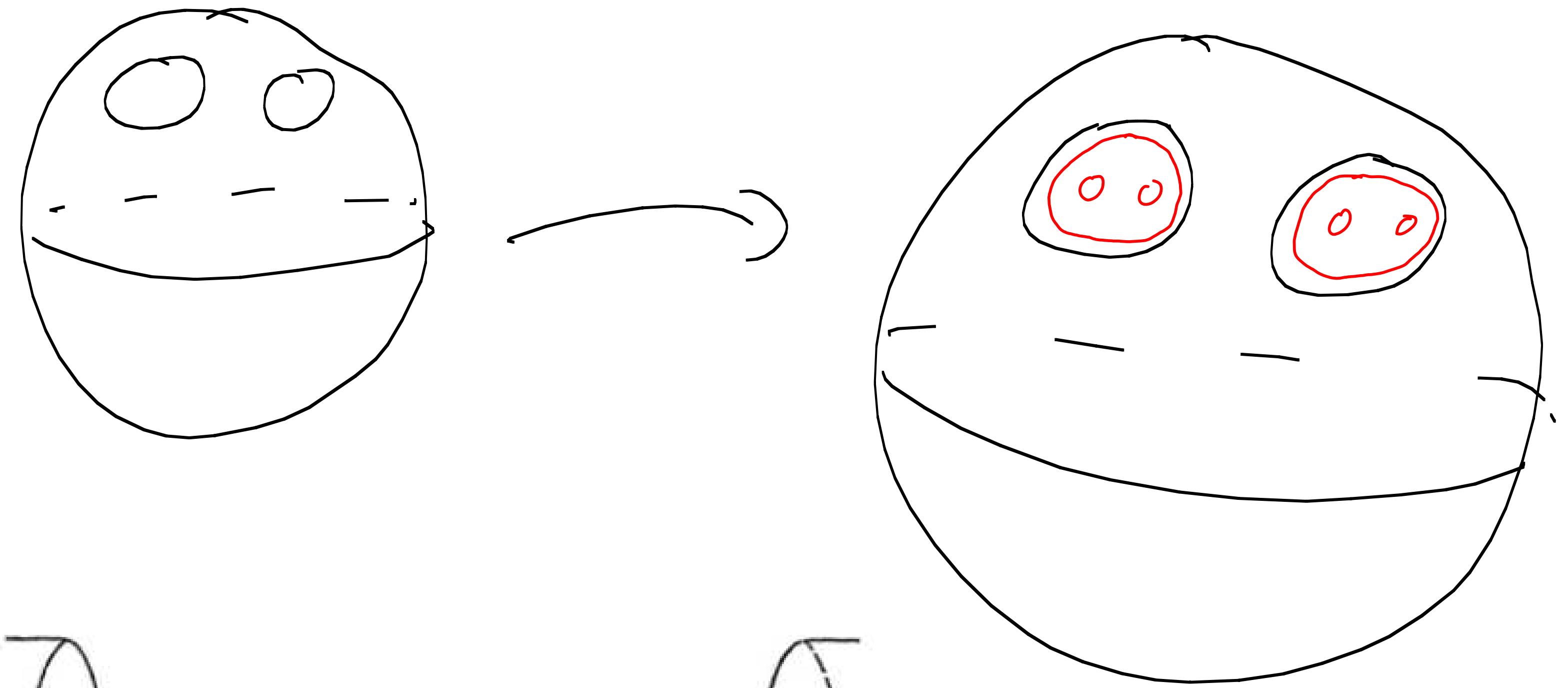


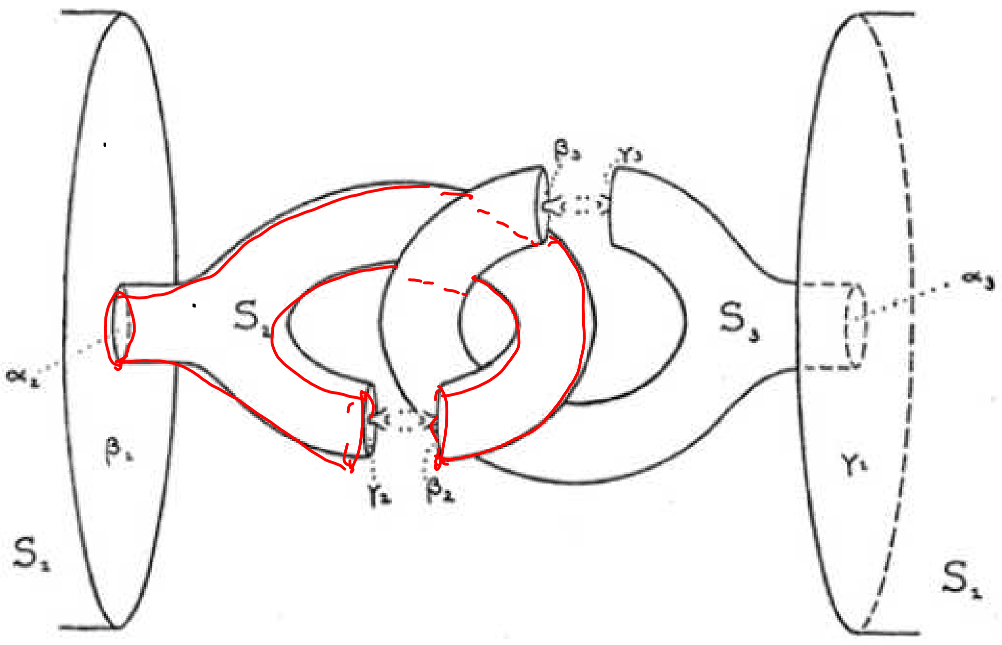




. The punctured tori with meridonal discs deleted are images of discs with two open subdiscs removed.

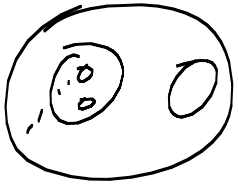
Thus, successive steps are images of:





The horned sphere as originally drawn by Alexander (1924) is illustrated above.

- Iterating, we get the image of $S^2 \setminus$ Cantor set
- As the discs get smaller, this extends to an embedding of S^2



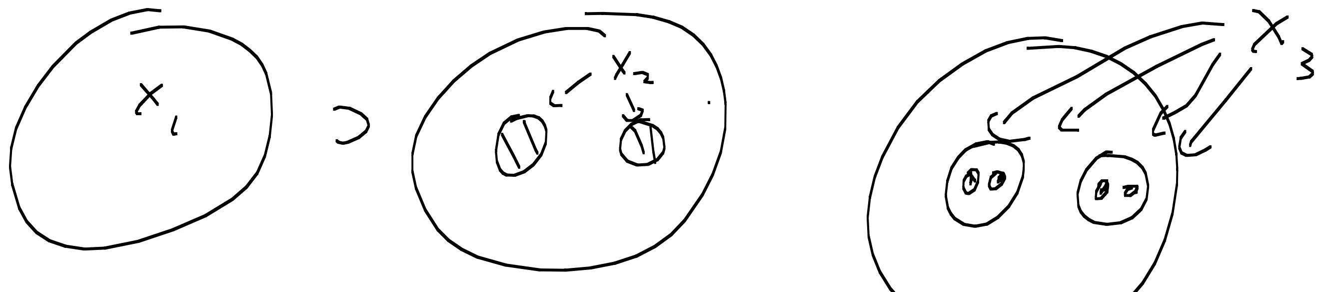
What is a Cantor Set?

Consider a metric space X s.t.

- X is compact
- X is totally disconnected, i.e., any connected subset of X is a single point
- Every point is a limit point. (Dense in itself)

Thm: X is homeomorphic to the Cantor set.

Example:

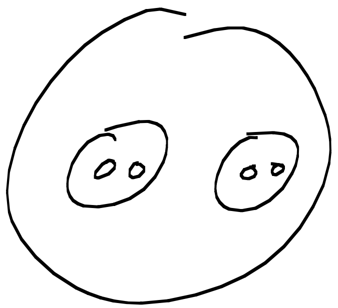


- Consider X_1 = Closed disc

X_i = 2 discs in each component of X_{i+1}

s.t. diam (Disc in X_i) $\longrightarrow 0$ as $i \longrightarrow \infty$

Claim: $X_\infty = \bigcap X_i$ = Cantor set.



Firstly · $X_\infty \subset X$ is closed

$\Rightarrow X_\infty$ is compact

· Next, X_∞ is totally disconnected,

as suppose $C \subset X_\infty$ is connected, and $C \neq \{pt\}$,

then $\text{diam}(C) > 0$

· Now, as $C \subset X_\infty \subset X_i$ is connected,

C is contained in some component of $X_i \; \forall i$.

· But diameter of components of $X_i \longrightarrow 0$

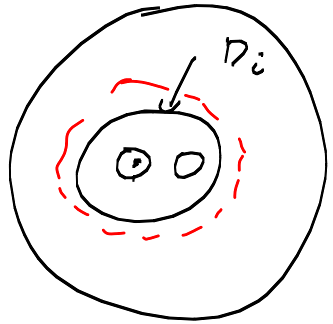
So far, we used:

$$\begin{cases} \cdot\ X_i \text{ compact} \\ \cdot\ \text{diameter of components of } X_i \to 0. \end{cases}$$

Dense in itself:

Suppose $x \in X_\infty$, we need to show

$\forall \varepsilon > 0, \ \exists \ x' \in B_{X_\infty}(x, \varepsilon), \ x' \neq x.$



- Now, if i is sufficiently large, the component D_i of X_i containing x is contained in $B_{X_\infty}(x, \varepsilon)$

- Choose $x' \in X_\infty$ in a different component of X_{i+1} which is contained in D_i.

We used: Each component of X_i contains at least two components of X_{i+1}.

Thus, given $X_\infty = \bigcap X_i$, $X_1 \supset X_2 \supset \dots$

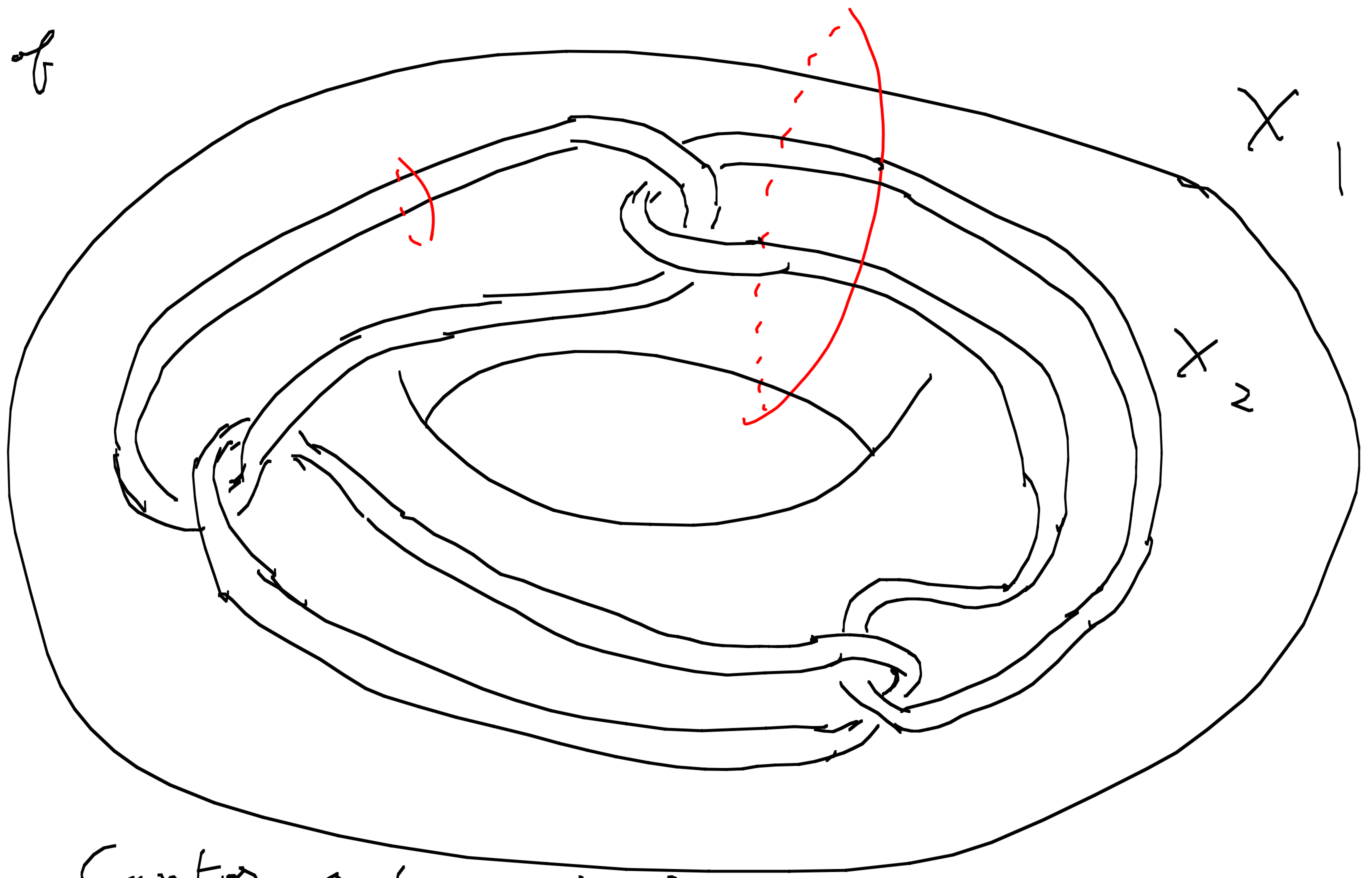
s.t.

- All X_i compact
- $\exists$ sequence $d_i \longrightarrow 0$ s.t. if $D_i \subset X_i$ connected, $\operatorname{diam}(D_i) \leq d_i$.
- If D_i is a component of X_i, $D_i \cap X_j$ has at least two components for some $j \geq i$.

<u>Exercise:</u> X_∞ is a Cantor set.

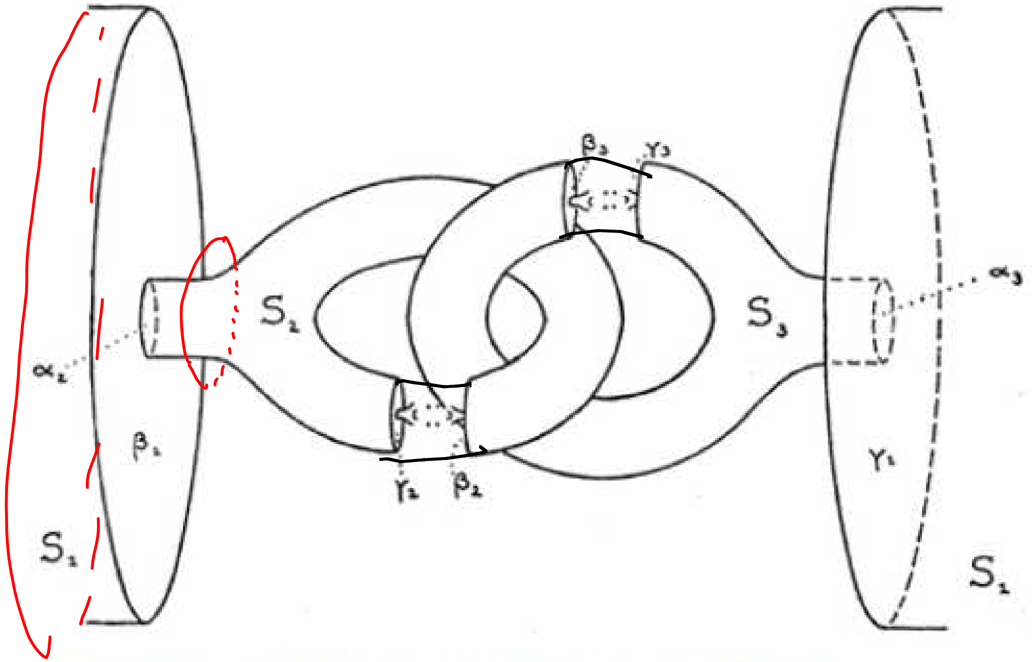
Example: Antoine's necklace.

- We consider a sequence of sets $X_i \subset S^3$ which satisfy the above. Let $X_\infty = \bigcap X_i$
- X_1 = Solid torus
- X_i = Union of solid tori



- X_∞ is a Cantor set in S^3
- $\pi_1(S^3 \setminus X_\infty) \neq 1$ (X_∞ is 'knotted')

Horned sphere complement



The horned sphere as originally drawn by Alexander (1924) is illustrated above.

Rk: Cantor set is $\prod_{i=1}^{\infty} \{0,1\}$ with the product topology.

- In the exterior of the horned sphere, the red curve above (meridian of the first torus) is not homotopically trivial.
- Hence the exterior is not a ball

Some remarks on the Horned sphere.

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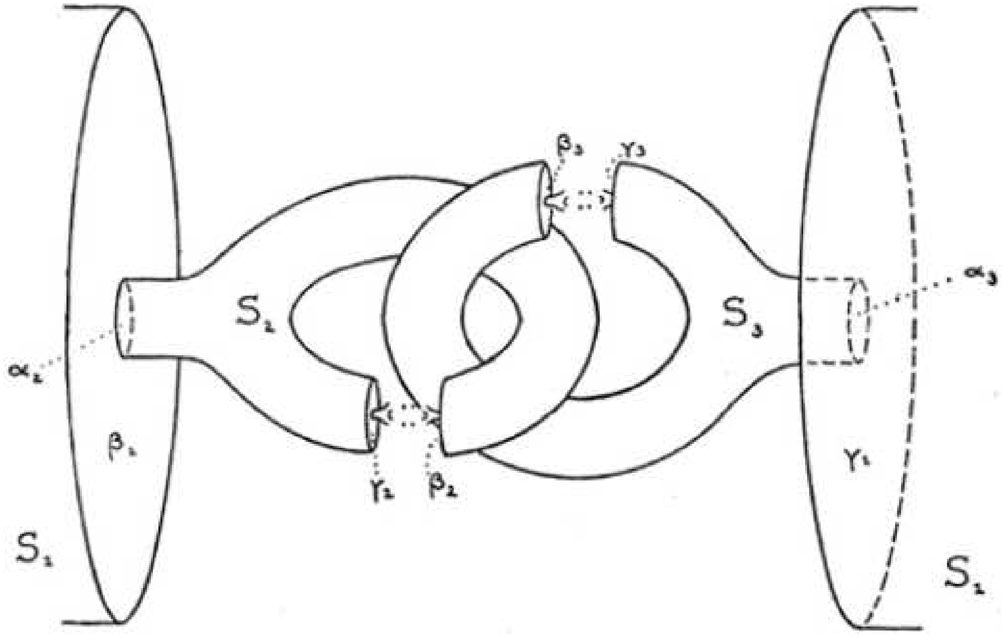

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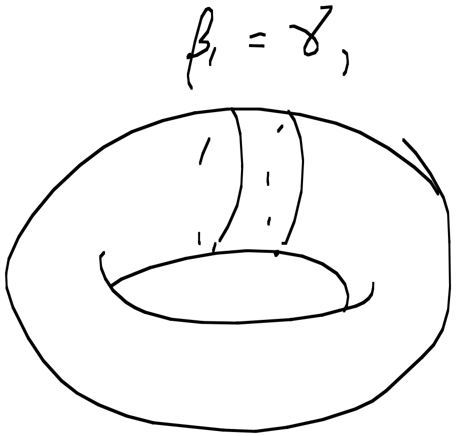
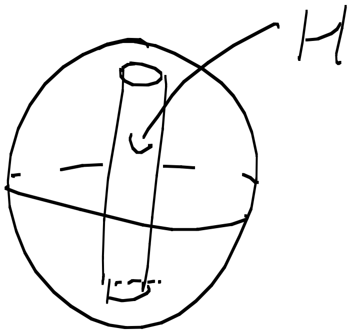
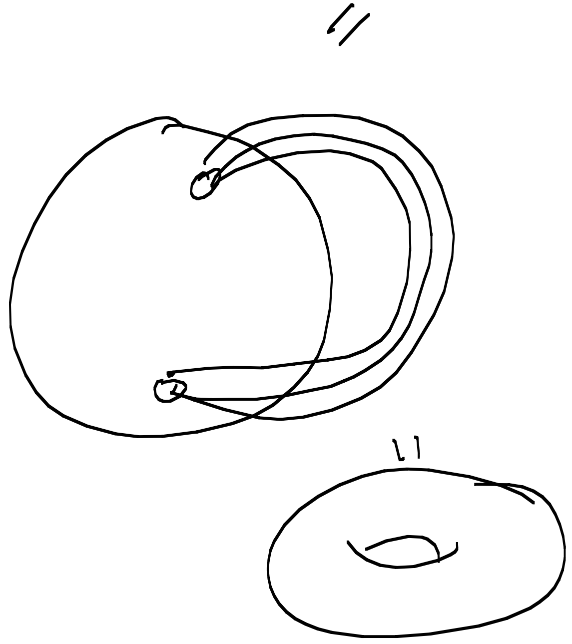


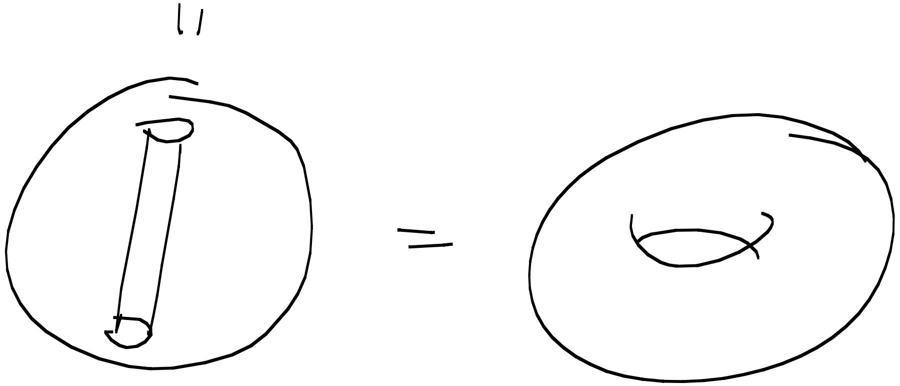
- The complement of the solid torus in S^3 = solid torus

$$S^3 = D^3_+ \cup D_3^- = (D^3_+ \cup H) \cup (D_3^- \setminus \mathring{H})$$



$$S^3 = (D^3_+ \cup H) \cup (D^3_- \setminus \mathring{H})$$

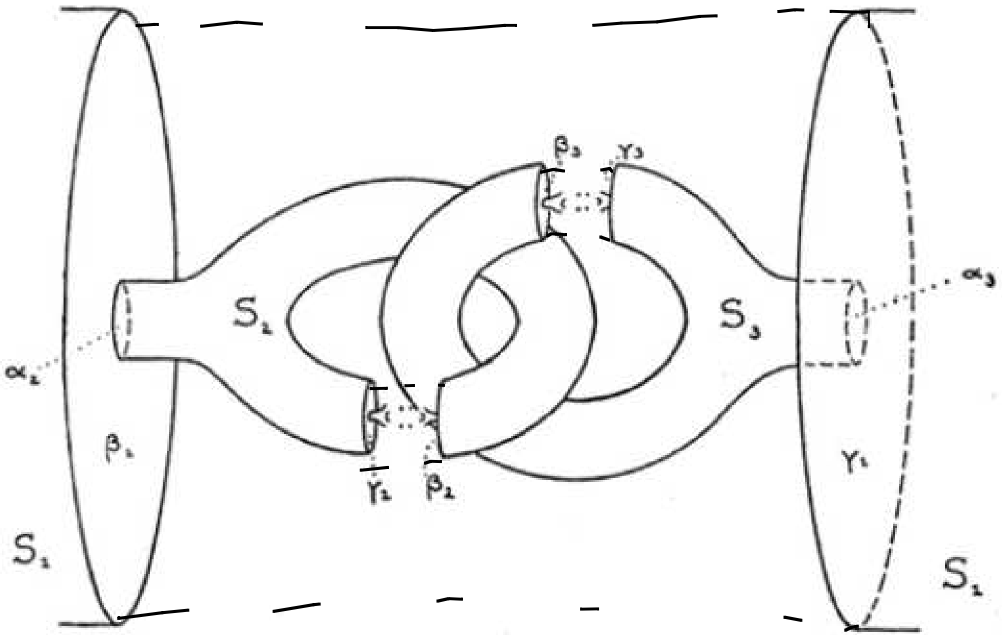




(standard solid torus)

$\cdot$ In particular, $\pi_1(S^3 \setminus \underset{\substack{\shortparallel \\ D^2 \times S^1 \overset{h.e.}{\sim} S^1}}{(D^2 \times S^1)}) = \mathbb{Z}$

$\therefore S^3 \setminus X_1 = D^2 \times S^1, \quad \beta_1 = \gamma_1 =$ generator of $\pi_1(S^3 \setminus X_1)$



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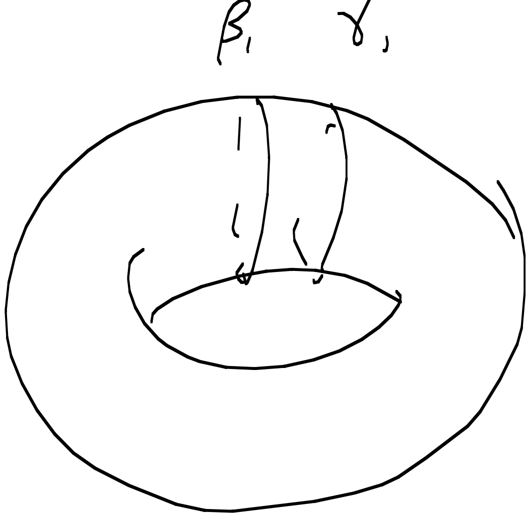
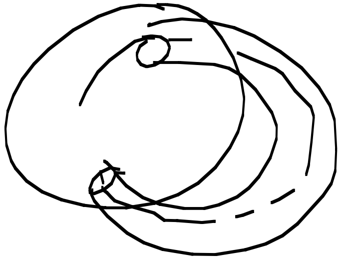
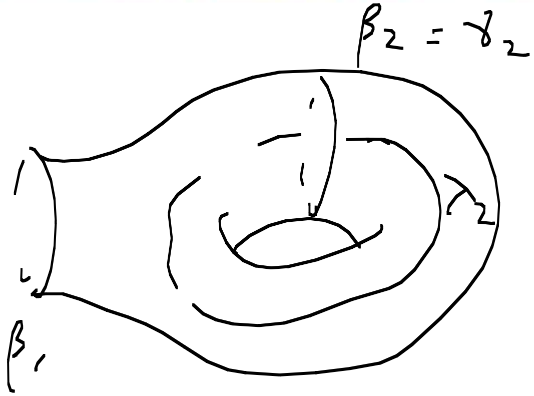






$, \quad \beta_1 = [\beta_2, \lambda_2]$

$\gamma_1 = [\beta_3, \lambda_3]$

Lower Central Series:

Let G be a group.

We inductively define

$$G_2 = [G, G]$$

$$G_{k+1} = [G_k, G], \quad k \geqslant 2$$

- Then G_k / G_{k+1} is abelian for all k, and each extension

$$1 \longrightarrow G_k \longrightarrow G_{k+1} \longrightarrow G_k / G_{k+1} \longrightarrow 1$$

is central. (G_k is central in G_{k+1}).

Showing $g \in G$ nontrivial (sometimes)

- Consider $[g] \in G/[G,G] = G_1/G_2$ – abelian group.
  - If $[g] \neq 0$, $g \neq 0$ $\{$We can verify this$\}$
  - If $[g] = 0$, then $g \in G_2$.
    - Consider $[g]$ in G_2/G_3 – again abelian.
- Iterating, we can check whether

  $g \in G_k \ \forall k$,

  i.e. $g \in G_\infty = \bigcap_{k=1}^{\infty} G_k$ $\}$

  - For horned sphere, if $G = \pi_1(S^3 \setminus X_k)$, then $\mu \notin G_{k+1}$

Theorem (Magnus): If F is a free group, $F_\infty = 1$.

CW-complexes & Cellular homology.

CW-complexes: Build up spaces by inductively attaching k-cells to the union of $(k-1)$-cells.

- Let X be a topological space.
- A CW-structure on X is a collection of maps $\{\varphi_\alpha : D_\alpha^k \longrightarrow X\}_{\alpha\in A}$ into X s.t.

(1) φ_α is injective on $\mathring{D}_\alpha$.

(2) $X = \coprod_{\alpha\in A} \varphi_\alpha(\mathring{D}_\alpha)$

(3) Let $X^{(k)} = \bigcup_{\substack{\alpha\in A \\ j=\dim(D_\alpha)\leq k}} \varphi_\alpha(\mathring{D}_\alpha^j)$. Then $\varphi_\alpha(\partial D_\alpha^k) \subset X^{(k-1)}$.

(4) The topology on X is the largest one such that φ_α are continuous, i.e.,

$$U \subset X \text{ is open iff } \forall \alpha,\ \varphi_\alpha^{-1}(U) \subset D_\alpha \text{ is open}$$

i.e., $f : X \to Y$ is continuous iff

$$\forall \alpha,\ f \circ \varphi_\alpha : D_\alpha \to Y \text{ is continuous.}$$

<u>Inductive description:</u> (assume finitely many cells)

- We describe inductively $X^{(k)}$.
- $X^{(0)}$ = Union of points with discrete topology.
- $X^{(k)} = X^{(k-1)} \cup \{k\text{-cells}\}$ is determined by

<u>attaching maps</u> $\Theta_\alpha : \underset{\cong \partial D_\alpha^k}{S_\alpha^{k-1}} \to X^{(k-1)}$

<u>Namely</u>:

- Let $X^{(k-1)}$ be a CW-complex with all cells of dimension $\leq k-1$.

- Given a collection of maps

$$\{\theta_\alpha : S_\alpha^{k-1} \to X^{(k-1)}\}_{\alpha \in A},$$

we define $X^{(k)}$ by

$$X^{(k)} = \left(X^{(k-1)} \sqcup \left(\bigsqcup_{\alpha \in A} D_\alpha^k \right) \right) \Big/ \sim$$

with $\underset{\partial D_\alpha^k = S_\alpha^{k-1}}{\underset{\cap}{x}} \sim \underset{X^{(k-1)}}{\underset{\cap}{\theta_\alpha(x)}}$ generating the equivalence relation $\sim$

Relation between the descriptions.

$$X^{(k)} = \left(X^{(k-1)} \amalg \left(\coprod_{\alpha \in A} D_\alpha^k \right) \right) \Big/ \sim$$

- Given a CW-complex constructed inductively using maps θ_α,

$$\varphi_\alpha : D_\alpha^k \to X^{(k)}$$ is the composition of inclusion into $X^{(k-1)} \amalg \amalg D_\alpha^k$ and the quotient map.

- Conversely, given maps φ_α, $\theta_\alpha = \varphi_\alpha |_{\partial D_\alpha^k = S_\alpha^{k-1}}$.
- $X = \bigcup_{k \geq 0} X^{(k)}$ (actually $\varinjlim X^{(k)}$)

Rk: We see inductively that $X = \coprod_{\alpha \in A} \varphi_\alpha(\mathring{D}_\alpha)$.

as $X^{(k)} = X^{(k-1)} \coprod \left(\amalg_{\alpha \in A} \mathring{D}_\alpha^k \right)$ as a set.

Cleaner description of inductive step.

Data: · A $\leq (k-1)$-dimensional CW-complex $X^{(k-1)}$.

· A collection of maps $\{\theta_\alpha : S^{k-1} \longrightarrow X^{(k-1)}\}_{\alpha \in A_k}$

Consider the space

$$X^{(k)} = \Big(X^{(k-1)} \coprod \overbrace{(D^k \times \underbrace{A_k}_{\substack{\uparrow \\ \text{discrete set}}})}^{\text{disjoint union}} \Big) \Big/ \sim$$

with $\sim$ the equivalence relation generated by:

for $(x, \alpha) \in (D^k \times A)$, if $x \in \partial D^k = S^{k-1}$,

$$(x, \alpha) \sim \theta_\alpha(x) \in X^{(k-1)}.$$

__For finite CW-complexes:__

- Given $X^{(k-1)}$, maps $\theta_1, \ldots, \theta_{n(k)} : S^{k-1} \to X^{(k-1)}$,

we let

$$X^{(k)} = \Big(X^{(k-1)} \amalg \big(\underbrace{D^k \amalg D^k \amalg \cdots \amalg D^k}_{n(k) \text{ copies}} \big) \Big) \Big/ \sim$$

so that for $x \in j^{th}$ copy of D^k,

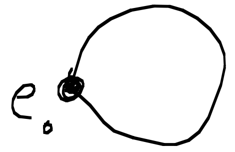
if $x \in \partial D^k$, $x \sim \theta_j(x) \in X^{(k-1)}$.

__Theorem (Whitehead):__ $X^{(k)}$ is determined up to homotopy by

- $X^{(k-1)}$ up to homotopy
- $\theta_\alpha : S^{k-1}_\alpha \to X^{(k-1)}$ __up to homotopy__.

<u>E.g.</u> . X has

- One 0-cell $\cdot\, e_0$
- One 1-cell, attaching map $\theta : \underset{\partial D^1}{\overset{\shortparallel}{S^0}} \to e_0$ constant.

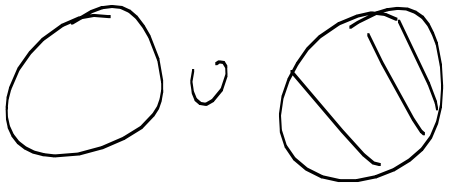


$= S^1 = e_0 \cup e_1$

- One 2-cell, attaching map

$$\theta : S^1 \to e_0 \cup e_1 = S^1$$

$$\theta(z) \mapsto z^2$$



$/\sim \; = S^1 \cup D^2/\sim, \; z \in \partial D^2 \sim z^2 \in S^1.$

$X = \mathbb{R}P^2$

More on $\mathbb{R}P^2$ example:

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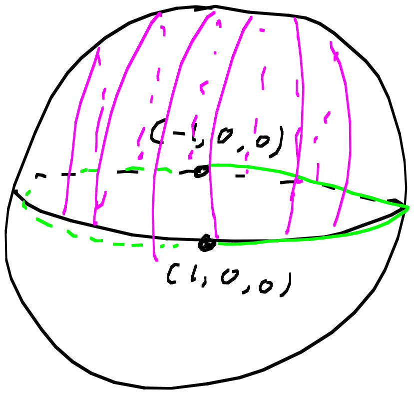
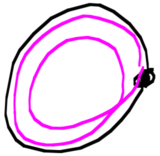
$$\mathbb{R}P^2 = S^2 / _{x \sim -x.} \quad , \ S^2 \subseteq \mathbb{R}^3$$





- We define a CW-structure using characteristic maps.

- <u>One 0-cell</u>: $\varphi_0 : \overset{\text{``}D^0}{\{0\}} \longmapsto [(1,0,0)] = [(-1,0,0)]$

- <u>One 1-cell</u>: $\varphi_1 : \overset{\text{``}D^1}{[0,1]} \longrightarrow S^2, \quad \varphi_1(t) = [(\cos(\pi t), \sin(\pi t), 0)]$

$$= [(-\cos(\pi t), -\sin(\pi t), 0)]$$

<u>Note</u>: $\varphi_1(\partial D^1) \subset \varphi_0(D^0)$

$\varphi_1|_{\text{int}(D^1)}$ is 1-1 and disjoint from $\varphi_0(\text{int}(D^0))$

- $\varphi_2 : D^2 \to$ Upper hemisphere | . $\varphi_2(\partial D^2) \subset X^{(1)}$ | as a set.

. $\mathbb{R}P^2 = \varphi_0(\mathring{D}^0) \sqcup \varphi_1(\mathring{D}^1) \sqcup \varphi_2(\mathring{D}^2)$

Thus,

- We have a collection of continuous maps φ_α
- There is a bijection from the union of the interiors of the cells to $\mathbb{RP}^2$.
- The topology induced by the CW-structure is compact and Hausdorff. (Ex).
- Hence we can show the given CW-complex is homeomorphic to $\mathbb{RP}^2$.
- Ex: Show this coincides with the description using Φ_α.

Homotopy type of CW-complexes

Let X be a topological space,

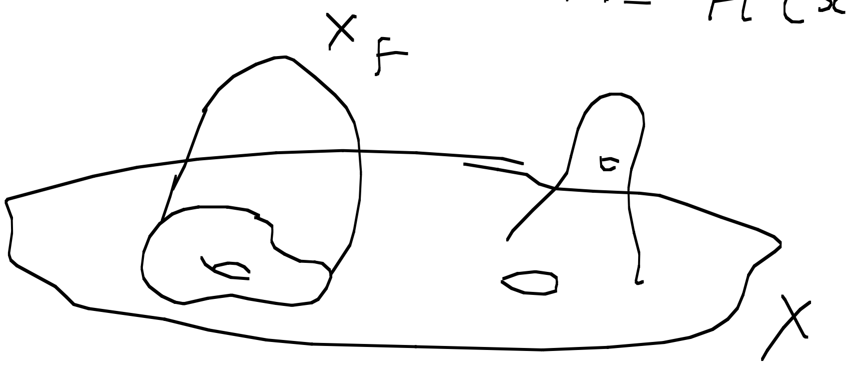
$f : S^{k-1} \longrightarrow X$ be a map

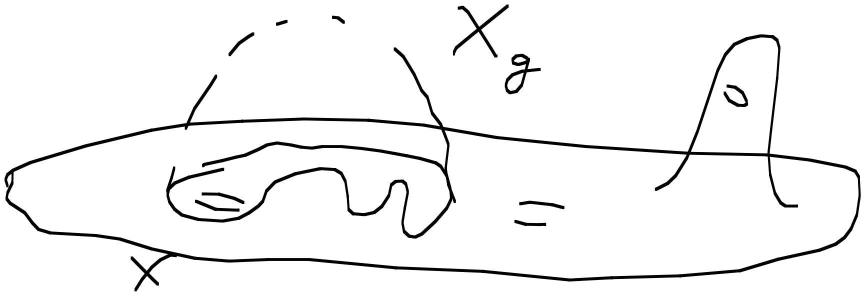
and $X_f = (X \amalg D^k)/_\sim$, with $\sim$ generated by

$$x \in \partial D^k \Rightarrow x \sim f(x).$$

Theorem: If $f \sim g : S^{k-1} \longrightarrow X$, then $X_f \overset{h.e.}{\simeq} X_g$.

Pf: Let $H : S^{k-1} \times [0,1] \rightarrow X$ be a homotopy from f to g.
$\overline{H}(x,t) := H(x, 1-t)$.



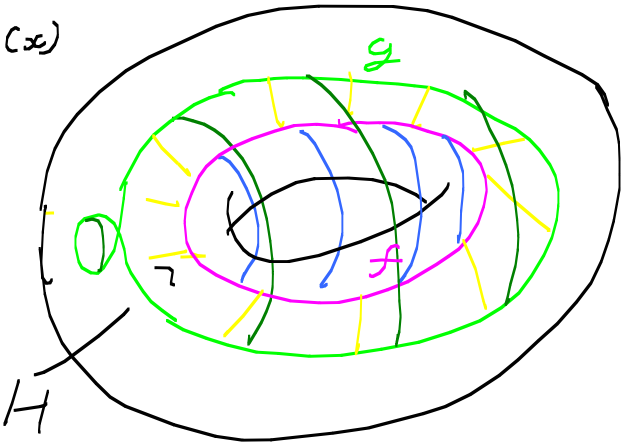


We first construct maps $F : X_f \longrightarrow X_g$ & $G : X_g \longrightarrow X_f$

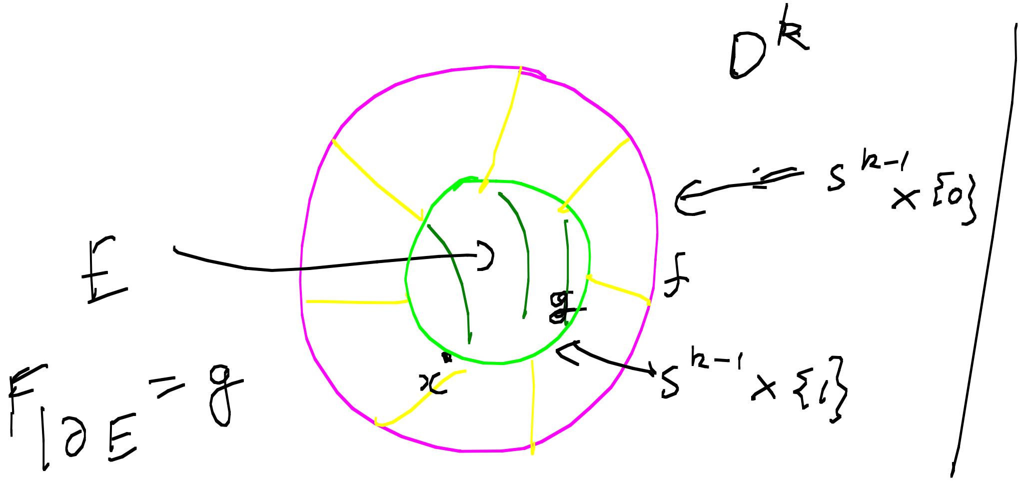
$$F : \overset{X_f}{\overbrace{(X \sqcup D^k)/_{\substack{x\sim f(x)\\ \partial D^2}}}} \longrightarrow \overset{= X_g}{(X \sqcup D^k)/_{\substack{x \sim g(x)\\ \partial D^2}}}$$

by

- $F_{|X} : X \longrightarrow X$ is the identity.



- On D^k we define F as follows:

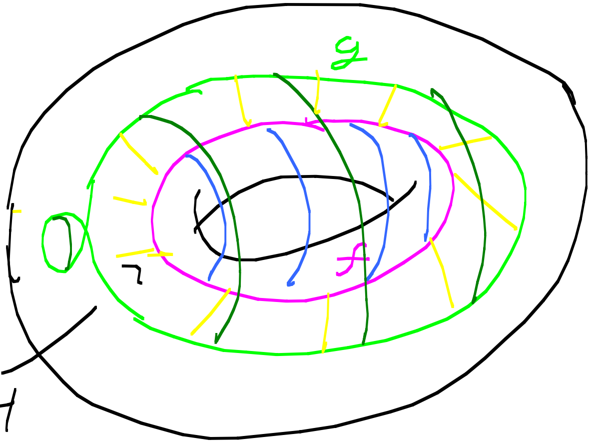


- If $x \in \partial D^2$, $F(x) = f(x)$
- Identify a nbd. A of ∂D^k with $S^{k-1} \times [0,1]$
- $F_{|A} = H : S^{k-1} \times [0,1] \longrightarrow X$
- $F_{|\overline{D^k \setminus A}}$ $\overset{= E}{}$ maps to $D^k \subset X_g$

- The map $G: X_g \longrightarrow X_f$ is constructed similarly.

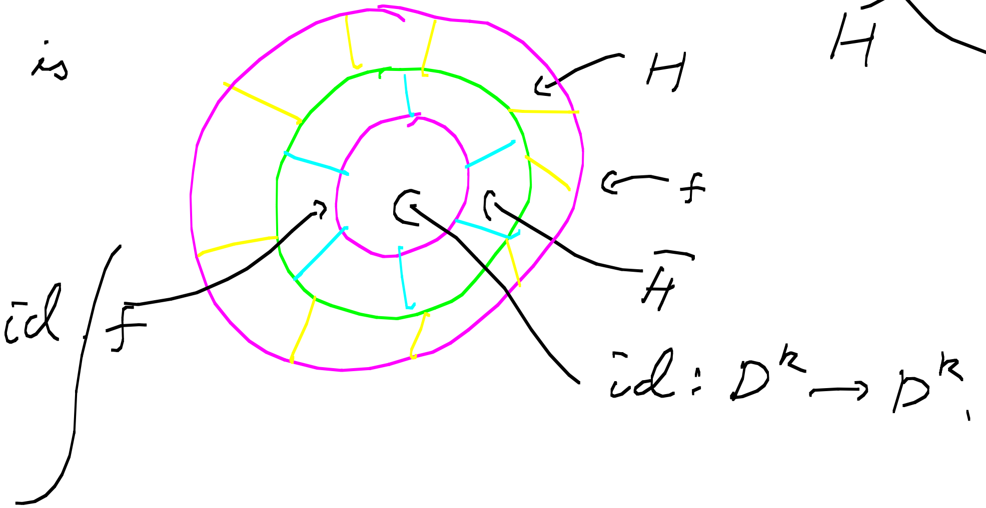
- **Lemma:** $G \circ F : X_f \longrightarrow X_f$ is homotopic to the identity.

Pf: $X_f = (X \amalg D^k) / \underset{\overset{\cap}{\partial D^k}}{x} \sim f(x)$



- $G \circ F_{|X}$ is the identity.

- $G \circ F_{|D^k}$ is



- Note $H \cdot \bar{H} \sim id$ just as $\alpha\bar{\alpha} = e$.

$\square$

Cellular homology

Let X be a CW-complex, $X^{(k)}$ - the k-skeleton.

- We define the chain complex $C_*^{CW}(X)$ by

$$C_k^{CW}(X) = \underbrace{H_k(X^{(k)}, X^{(k-1)})}_{\text{singular homology}}$$

Rk: This is the free abelian group with basis k-cells

- The boundary map

$$d_k : H_k(X^{(k)}, X^{(k-1)}) \to H_{k-1}(X^{(k-1)}, X^{(k-2)})$$

is the composition

$$H_k(X^{(k)}, X^{(k-1)}) \xrightarrow{\partial} H_{k-1}(X^{(k-1)}) \xrightarrow{j_*} H_{k-1}(X^{(k-1)}, X^{(k-2)})$$

These come from

$$\begin{tikzcd}
{} \arrow[r] & H_k(X^{(k)}, X^{(k-1)}) \arrow[r, "\partial"] & H_{k-1}(X^{(k-1)}) \arrow[r] \arrow[dr, equal, "\sim"] & H_{k-1}(X^{(k)}) & \\
& {} \arrow[r] & H_{k-1}(X^{(k-2)}) \arrow[r] & H_{k-1}(X^{(k-1)}) \arrow[r] & H_{k-1}(X^{(k-1)}, X^{(k-2)})
\end{tikzcd}$$

RK: In general, we can consider any filtration $X^{(n)}$. Then $H_k(X^{(n)}, X^{(n-1)})$ forms a bigraded complex.

- We have boundary maps $H_k(X^{(n)}, X^{(n-1)}) \to H_{k-1}(X^{(n-1)}, X^{(n-2)})$

i.e.
$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & C_k(X^{(n)}) \arrow[r] \arrow[d] & C_k(X^{(n+1)}) \arrow[d] \\
{} \arrow[r] & C_{k-1}(X^{(n)}) \arrow[r] \arrow[d] & C_{k-1}(X^{(n+1)}) \arrow[d] \\
& {} & {}
\end{tikzcd}$$

$\xrightarrow{\text{gives}}$ Maps on homology, also bigraded

The cellular boundary map

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$C_k^{CW}(X) = H_k(X^{(k)}, X^{(k-1)})$, $d_k : C_k^{CW}(X) \to C_{k-1}^{CW}(X)$ comes from:

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(X^{(k)}, X^{(k-1)}) \arrow[r, "\partial_k"] & H_{k-1}(X^{(k-1)}) \arrow[r] \arrow[d, equal] & \cdots & & \\
& \cdots \arrow[r] & H_{k-1}(X^{(k-1)}) \arrow[r, "j_{k-1}"'] \arrow[d, equal] & H_{k-1}(X^{(k-1)}, X^{(k-2)}) \arrow[r, "\partial_{k-1}"] & H_{k-2}(X^{(k-2)}) \arrow[r] \arrow[dl, "="] & \cdots \\
& & (X^{(k-1)}, \emptyset) & H_{k-2}(X^{(k-2)}) \arrow[r, "j_{k-2}"] & H_{k-2}(X^{(k-2)}, X^{(k-3)}) &
\end{tikzcd}$$

$[d_k = j_{k-1} \circ \partial_k]$

Lemma: $d_{k-1} \circ d_k = 0$

Pf: Look at the above diagram.

$$d_{k-1} \circ d_k = j_{k-2} \circ \underset{\overset{\shortparallel}{0}}{(\partial_{k-1} \circ j_{k-1})} \circ \partial_k .$$

Thus, we can define

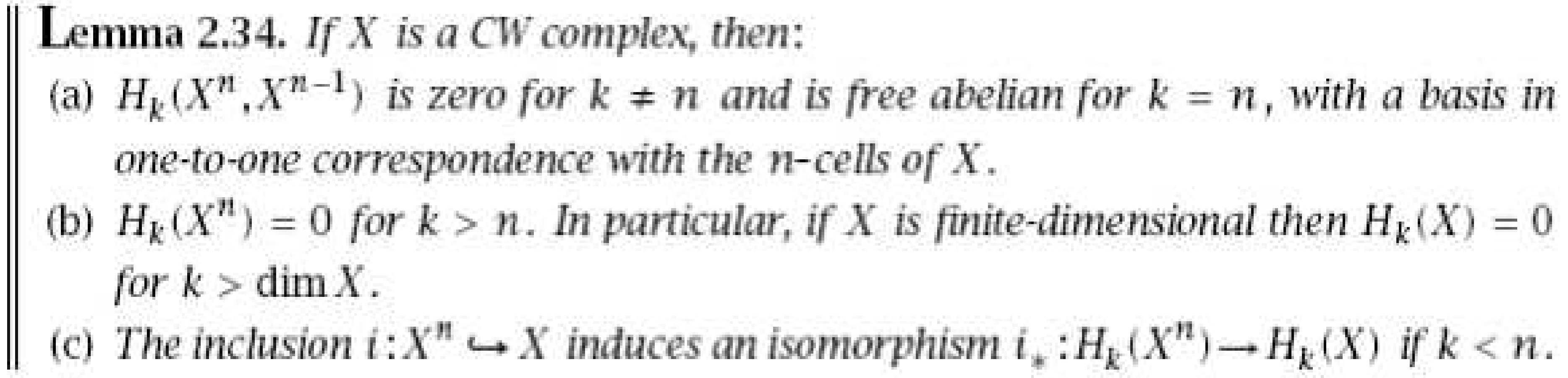
$$H_*^{CW}(X) = H_*(C_*^{CW}, d_*).$$

Theorem: $H_*(X) \cong H_*^{CW}(X)$. (uses lemma from below)

Pf: Consider the commutative diagram: (diagonals exact)

$$\begin{tikzcd}
 & & & & 0 = H_n(X^{n+1}, X^n) & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

We use the lemma:



Lemma 2.34. *If X is a CW complex, then:*

(a) $H_k(X^n, X^{n-1})$ *is zero for* $k \neq n$ *and is free abelian for* $k = n$, *with a basis in one-to-one correspondence with the* n-*cells of* X.

(b) $H_k(X^n) = 0$ *for* $k > n$. *In particular, if* X *is finite-dimensional then* $H_k(X) = 0$ *for* $k > \dim X$.

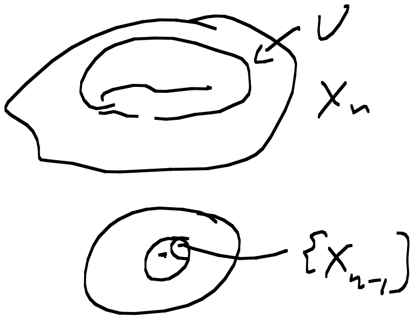
(c) *The inclusion* $i : X^n \hookrightarrow X$ *induces an isomorphism* $i_* : H_k(X^n) \to H_k(X)$ *if* $k < n$.

Pf: (a). First, we see that $H_k(X^{(n)}, X^{(n-1)}) = \tilde{H}_k(X^{(n)}/X^{(n-1)})$, where A/B is the quotient of A with B identified to a point.

· Note that $X^{(n-1)} \subset U$, a nbd. that defm. retracts to $X^{(n-1)}$.

[$H_k(X, A) = \tilde{H}_k(X/A)$ if $\bar{A} \subset U$ open, U defm. retracts to A]

Hence,



$$H_*(X^{(n)}, X^{(n-1)}) = H_*(X^{(n)}, U)$$

$$\overset{\text{Excision}}{=} H_*(X^{(n)} \setminus X^{(n-1)}, U \setminus X^{(n-1)})$$

$$\overset{\text{Same spaces}}{=} H_*\left(X^{(n)}/X^{(n-1)} - \{x^{(n-1)}\}, U/X^{(n-1)} \setminus \{x^{(n-1)}\}\right)$$

$$\overset{\text{Excision}}{=} H_*(X^{(n)}/X^{(n-1)}, U/X^{(n-1)})$$

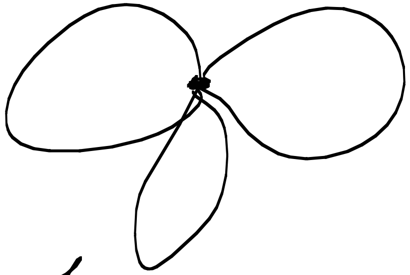
Now, as U defm. retracts to $X^{(n-1)}$, $\tilde{H}_*(U/X^{(n-1)}) = 0$

$\therefore$

$$\to \overset{\overset{0}{\shortparallel}}{\tilde{H}_*(U/X^{(n-1)})} \to \tilde{H}_*(X^{(n)}/X^{(n-1)}) \to H_*(X^{(n)}/X^{(n-1)}, U/X^{(n-1)}) \to \overset{\overset{0}{\shortparallel}}{\tilde{H}_{*-1}(U/X^{(n-1)})}$$

$$\therefore \quad H_*(X^{(n)}, X^{(n-1)}) = \tilde{H}_*(X^{(n)}/X^{(n-1)})$$

Next, observe that $X^{(n)}/X^{(n-1)}$ is a wedge of spheres, with one sphere corresponding to each n-cell (Exercise)



(Wedge of spaces: Disjoint union of Spaces with a given basepoint on each of them identified)

Finally, we can compute the homology of a wedge of spheres using Mayer-Vietoris (and compact support if there are infinitely many.

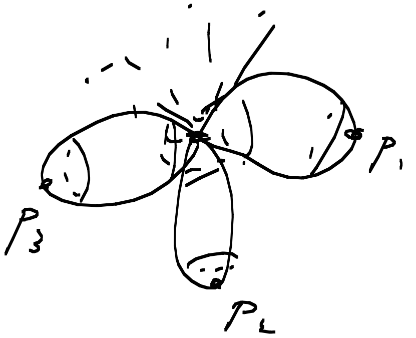
(Ex)

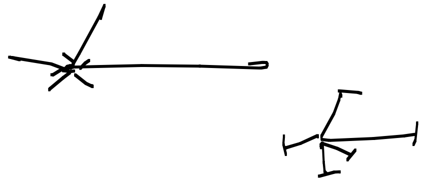
Using compact support

First we note the following lemma

Lemma: If $K \subset \bigvee_{\alpha\in A} S^n_\alpha$ (wedge of spheres) is compact, then K is disjoint from the 'opposite poles' p_α of all but finitely many spheres S^n_α

Pf: Note that otherwise $K \cap \left(\bigcup_{\alpha\in A}\{p_\alpha\}\right)$ is a discrete infinite set without limit points.





Next: If $\alpha_1, \dots, \alpha_n \in A$, then

$$S_{\alpha_1} \vee S_{\alpha_2} \vee \dots \vee S_{\alpha_n} \bigvee_{\substack{\alpha\in A \\ \alpha\neq\alpha_1,\dots,\alpha_n}} (S_\alpha \setminus \{p_\alpha\})$$ defm. retracts to

$S_{\alpha_1} \vee S_{\alpha_2} \vee \dots \vee S_{\alpha_n}$.

Next, we see

Lemma: $H_*(\bigvee_{\alpha \in A} S^n) \xleftarrow{\cong} \bigoplus_{\alpha \in A} H_*(S^n_\alpha)$

Pf: We have a map induced by inclusions.

Surjectivity: If $\sigma \in H_k(\bigvee_{\alpha \in A} S^n)$, then

$\exists K$ cpt. s.t. $\sigma \in H_k(K)$

· By the above, $\sigma \in H_k(S_{\alpha_1} \vee S_{\alpha_2} \vee \dots \vee S_{\alpha_n} \bigvee_{\substack{\alpha \in A \\ \alpha \neq \alpha_1 .. \alpha_n}} (S_\alpha \setminus \{p_\alpha\}))$

$$\begin{aligned} &\qquad\qquad \shortparallel \\ &H_k(S_{\alpha_1} \vee \dots \vee S_{\alpha_n}) \subset \bigoplus_{\alpha \in A} H_*(S^n_\alpha) \end{aligned}$$

Injectivity: If $\sigma = \underset{\substack{\cap \\ S_{\alpha_1}}}{\sigma_1} + \dots + \underset{\substack{\cap \\ S_{\alpha_n}}}{\sigma_n} = 0$, then $\sigma = 0$ in $H_{\cancel{k}}(K)$

for K cpt. $\Rightarrow$ $\sigma = 0$ in $H_k(S^n_{\alpha_1} \vee \dots \vee S^n_{\alpha_n} \vee S^n_{\alpha_{n+1}} \dots \vee S^n_{\alpha_N}) \Rightarrow \dots$

Thus, we have shown:

$$H_*(X^{(n)}, X^{(n-1)}) = \tilde{H}_*(X^{(n)}/X^{(n-1)}) = \tilde{H}_*(\bigvee_{\underset{\text{set of } n\text{-cells}}{\overset{\alpha\in A}{\shortparallel}}} S^{\alpha})$$

$$= \bigoplus_{\alpha\in A} \tilde{H}_*(S^{\alpha})$$

$$H_k(X^{(n)}, X^{(n-1)}) = \begin{cases} \bigoplus_{\alpha\in A} \mathbb{Z}, & k=n \\ 0 & \text{otherwise} \end{cases}$$

(b) $H_k(X^{(n)}) = 0$ for $\underline{\underline{k>n}}$ follows inductively on n by

$$-- \to \underset{\underset{0}{\shortparallel}}{H_k(X^{(n-1)})} \to H_k(X^{(n)}) \to \underset{\underset{0}{\shortparallel}}{H_k(X^{(n)}, X^{(n-1)})} \to --$$

(c) $i : X^{(n)} \longrightarrow X$ induces an isomorphism

$$H_k(X^{(n)}) \xrightarrow{i_*} H_k(X)$$

if $k < n$.

Pf: We first show:

Lemma: If $m \geq n$, $i_* : H_k(X^{(n)}) \xrightarrow{\simeq} H_k(X^{(m)})$.

Pf: By induction on m. If $m = n$, this is obvious.

- Now, if it holds for m

$$\underset{\underset{0}{\shortparallel}}{H_{k+1}(X^{(m+1)}, X^{(m)})} \longrightarrow H_k(X^{(m)}) \longrightarrow H_k(X^{(m+1)}) \longrightarrow \underset{\underset{0}{\shortparallel}}{H_k(X^{(m+1)}, X^{(m)})}$$

(as $k+1 \leq n < m+1$.

- Hence $H_k(X^{(m)}) \simeq H_k(X^{(m+1)})$. Use induction hypothesis.

- Rest of (c): Compact support.

Thm: $H_n^{cw}(X) = H_n(X)$

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Pf: We use the commutative diagram (follows from lemma)

$$\begin{tikzcd}
 & & & & & 0 = H_n(X^{n+1}, X^n) & & \\
 & & H_n(X^{n-1}) = 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & \text{(c) of lemma.} \arrow[l] & & \\
 & & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[rr] & & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] \arrow[ur] & \cdots \\
 & \cdots \arrow[ur] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & & & & 0 = H_{n-1}(X^{n-2}) \; [\text{(b) of lemma}] \arrow[ur] & & &
\end{tikzcd}$$

Hence: $H_n(X) \overset{= H_n(X^{n+1})}{\cong} \frac{H_n(X^n)}{\partial_{n+1}(H_{n+1}(X^{n+1}, X^n))} \underset{\substack{\uparrow \\ \text{as } j_n \\ \text{is } 1-1}}{=} \frac{j_n(H_n(X^n))}{j_n \circ \partial_{n+1}(H_n(X^{n+1}, X^n))} = \frac{\overset{= \ker \partial_n}{\ker d_n}}{\operatorname{im} d_{n+1}}$

$= H_n^{cw}(X)$

∎

Ex: More transparent pf using induction, Mayer-Vietoris.

- Assume CW-complexes are finite.

Effect of adding an n-cell: e_n - n-cell

$\partial e_n \subset X^{(n-1)}$ gives a homology class

Using Mayer-Vietoris:

- Singular Homology is unchanged except H_{n-1} & H_n
- Change in H_{n-1} : $[\partial e_n] \in H_{n-1}(X^{(n-1)})$ becomes 0,
  i.e., we quotient by this.
- Change in H_n : If $k \cdot [\partial e_n] = 0$, $k e_n$ gives
  a cycle.

Cellular homology: This changes the same way as singular homology.

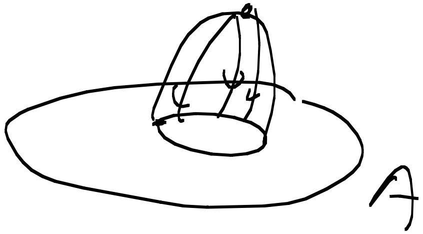
Attaching an n-cell in homology

Let A be a space, $\theta : \underset{\partial D^n}{\underset{\shortparallel}{S^{n-1}}} \to A$ a map

and $X = A \amalg D^n \big/ {}_{x \in \partial D^n \sim \theta(x)}$

Homology of X: Let $V = X \setminus \underset{\mathring{D}^n}{\underset{\cap}{\{0\}}}$: defm retracts to A

$X = V \cup \mathring{D}^n,$ $\quad V \cap \mathring{D}^n \underset{\text{h.e.}}{\simeq} S^{n-1})$



Mayer - Vietoris:

$$\to \tilde{H}_k(V \cap \mathring{D}^n) \to \tilde{H}_k(V) \oplus \overset{=0}{\tilde{H}_k(\mathring{D}_n)} \to \tilde{H}_k(X) \to H_k(V \cap \mathring{D}_n) \to$$

i.e.,

$$\to \tilde{H}_k(S^{n-1}) \to \tilde{H}_k(A) \underset{\cong \text{ if } k \neq n, n-1}{\xrightarrow{\uparrow}} \tilde{H}_k(X) \to \tilde{H}_{k-1}(S^{n-1}) \to \tilde{H}_{k-1}(A) \to \ldots$$

Using:

$$\to \tilde{H}_k(S^{n-1}) \longrightarrow \tilde{H}_k(A) \underset{\substack{\uparrow \\ \simeq \text{ if } k \neq n, n-1}}{\longrightarrow} \tilde{H}_k(X) \longrightarrow \tilde{H}_{k-1}(S^{n-1}) \longrightarrow \tilde{H}_{k-1}(A) \to \dots$$

- If $k \neq n, n-1$, $\tilde{H}_k(A) \simeq \tilde{H}_k(X)$

- If $k = n-1$, we get

$$\tilde{H}_{n-1}(S^{n-1}) \xrightarrow{\theta_*} \tilde{H}_{n-1}(A) \longrightarrow \tilde{H}_{n-1}(X) \longrightarrow 0$$

$$\Rightarrow \tilde{H}_{n-1}(X) = \tilde{H}_{n-1}(A) \big/ \operatorname{im}(\theta_*) .$$

- If $k = n$, we get

$$0 \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \underset{\substack{\mathbb{Z} \\ \wr\wr}}{H_{n-1}(S^{n-1})} \xrightarrow{\theta_*} H_{n-1}(A) \longrightarrow \dots$$

Hence $0 \to \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \ker(\theta_*) \to 0.$

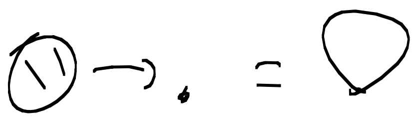
$$\Longrightarrow \tilde{H}_n(X) = \tilde{H}_n(A) \oplus \ker(\theta_*)^{\simeq m\mathbb{Z},\ m \geq 0}$$

Application:

. If X is a CW-complex with cells only in even dimension $n = 0, 2, 4, \dots$, then $H_k(X)$ is the free abelian group generated by k-cells.

Rk: This is quite common for complex manifolds.

E.g. $S^2 = \mathbb{C}P^1$: This has a CW-complex with 1 0-cell, 1 2-cell



E.g. $$\mathbb{C}P^2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0,0,0\}\} / (z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3)$$

$$= \underset{\substack{\cong \\ \mathbb{C}^2 = \mathring{D}^4}}{\{[z_1, z_2, 1] : z_1, z_2 \in \mathbb{C}^2\}} \cup \underset{\substack{\cong \\ \mathbb{C}P^1 = \mathring{D}^2 \cup \mathring{D}^0}}{\{(z_1, z_2, 0) : (z_1, z_2) \neq (0,0)\} / \sim}$$

Cell structure for $\mathbb{C}P^2$ ($\mathbb{C}P^n$, $\mathbb{R}P^n$ similar)

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$$\mathbb{C}P^2 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : (z_0, z_1, z_2) \neq (0,0,0)\} / \sim$$

with $(z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda z_2)$, $\lambda \neq 0$, $\lambda \in \mathbb{C}$.

- The class of (z_0, z_1, z_2) is denoted $[z_0 : z_1 : z_2]$.

- $\mathbb{C}P^1 \subset \mathbb{C}P^2$, $[z_0 : z_1] \longmapsto [z_0 : z_1 : 0]$.
- $\mathbb{C}P^2 \setminus \mathbb{C}P^1 = \{[z_0 : z_1 : z_2] : z_2 \neq 0\} = \{[z_0 : z_1 : 1]\} \cong \underset{\underset{\mathring{B}^4}{\|}}{\mathbb{C}^2}$.

<u>Inductively:</u> $\mathbb{C}P^1$ is a CW-complex with 1 0-cell, 1 2-cell.

- We have a map $\varphi : \underset{\underset{\mathbb{C}^2}{\cap}}{B^4} \longrightarrow \mathbb{C}P^2$, $\varphi(\partial B^4) \subseteq \mathbb{C}P^1 \cong \left\{\begin{matrix} \text{0-cell} \\ \cup \text{ 2-cell} \end{matrix}\right\}$

given by $(z_0, z_1) \longmapsto [z_0 : z_1 : \sqrt{1 - |z_0|^2 - |z_1|^2}]$

- This is 1-1 on $\mathring{B}^4$ & continuous. (Ex)

<u>On boundary</u>: $(z_0, z_1) \in S^3$, i.e. $\|z_0\|^2 + \|z_1\|^2 = 1$.

$\varphi : (z_0, z_1) \mapsto [z_0 : z_1 : 0] \in \mathbb{C}P^1 = S^2$

Now, if $|\lambda| = 1$, $\lambda \in \mathbb{C}$,

$(\lambda z_0, \lambda z_1) \in \partial B^4$,

$$\varphi(\lambda z_0, \lambda z_1) = [\lambda z_0, \lambda z_1 : 0] = [z_0 : z_1 : 0] = \varphi(z_0, z_1).$$

Thus, we have a map $\varphi : S^3 \longrightarrow S^2$ with the inverse image of each $p \in S^2$ a circle.

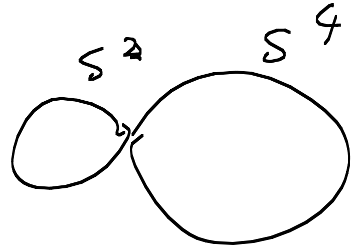
$$\left(\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}\right)$$

- This is called the <u>Hopf fibration</u>.
- This is a homotopically non-trivial map and in fact generates $\pi_3(S^2) = \mathbb{Z}$.

Chain complex for $\mathbb{C}P^2$ (cellular homology)

For $\mathbb{C}P^2$

$$\begin{tikzcd}
C_4 \arrow[d] & = & \mathbb{Z} \arrow[d] \\
C_3 \arrow[d] & = & 0 \arrow[d] \\
C_2 \arrow[d] & = & \mathbb{Z} \arrow[d] \\
C_1 \arrow[d] & = & 0 \arrow[d] \\
C_0 & = & \mathbb{Z}
\end{tikzcd}$$



For $S^2 \vee S^4$, this is a CW-complex obtained from S^2 by attaching a 4-cell with attaching map constant.

Cellular chain complex:

- Same as $\mathbb{C}P^2$
- Hence homology is the same.

Hence, $H_k(\mathbb{C}P^2) = \begin{cases} \mathbb{Z}, & k = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases}$

__Qn:__ Are $\mathbb{C}P^2$ and $S^2 \vee S^4$ homotopically equivalent?

- Homology, and even chain complexes, cannot detect the difference.

__Ans:__ They are different using 'cup product' on __cohomology__.

Cochains & cohomology.

- A cochain complex (C^*, δ^*) is a collection of graded free abelian groups (or free R-modules) $C^0, C^1, C^2, \ldots$, together with $\delta^k : C^k \to C^{k+1}$ s.t. $\delta \circ \delta = 0$

- Given a cochain complex (C^*, δ^*), its cohomology groups are

$$H^k = \ker(\delta^k) / \operatorname{Im}(\delta^{k-1}) \qquad \left(\delta^k \circ \delta^{k-1} = 0 \Rightarrow \ker(\delta^k) \supset \operatorname{Im}(\delta^{k-1}) \right)$$

Cochains from chains:

Let (C_*, ∂_*) be a chain complex and let A be an abelian group.

Let $C^k = \operatorname{Hom}(C_k, A)$,

$\delta^k : \operatorname{Hom}(C_k, A) \longrightarrow \operatorname{Hom}(C_{k+1}, A)$

be $\delta^k(\varphi) = \varphi \circ \partial_{k+1}$, i.e. δ^k is the dual of ∂_{k+1}.

- As $\partial \circ \partial = 0$, $\delta \circ \delta = 0$.

Thus, we can define singular, simplicial, cellular coChain complexes of a space X; $H^k(X, A)$

- The cohomology of these gives the cohomology of X.

Universal coefficients theorem: If (C_*, ∂_*) & (C'_*, ∂'_*) are chain complexes with isomorphic homology groups (with $\mathbb{Z}$-coefficients), then for every abelian group A, $H^k(C_*, A) = H^k(C'_*, A) \, \forall k$.

Examples: Chain complex of lens spaces (includes $\mathbb{RP}^3$).

$L = L(p,q)$:

$$\begin{tikzcd}
C_3 \arrow[d] & = & \mathbb{Z} \arrow[d, "0"] \\
C_2 \arrow[d] & = & \mathbb{Z} \arrow[d, "\times p"] \\
C_1 \arrow[d] & = & \mathbb{Z} \arrow[d, "0"] \\
C_0 & = & \mathbb{Z}
\end{tikzcd}$$

Homology :
$$\begin{cases} H_0 = \mathbb{Z} \\ H_1 = \mathbb{Z}/p\mathbb{Z} \\ H_2 = 0 \\ H_3 = \mathbb{Z} \end{cases}$$

Cohomology with coefficients $\mathbb{Z}$.

$$\begin{tikzcd}
C^3 & = & \mathbb{Z} \\
C^2 \arrow[u] & = & \mathbb{Z} \arrow[u, "0"'] \\
C^1 \arrow[u] & = & \mathbb{Z} \arrow[u, "p"'] \\
C^0 \arrow[u] & = & \mathbb{Z} \arrow[u, "0"']
\end{tikzcd}$$

$$\begin{cases} H^0(L,\mathbb{Z}) = \mathbb{Z} \\ H^1(L,\mathbb{Z}) = 0 \\ H^2(L,\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} \\ H^3(L,\mathbb{Z}) = \mathbb{Z} \end{cases}$$

Thm: $H^1(X,\mathbb{Z}) = \mathrm{Hom}(H_1(X),\mathbb{Z})$.

Cohomology with coefficient $\mathbb{Z}/p\mathbb{Z}$: $C^k(L, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}(C_k, \mathbb{Z}/p\mathbb{Z})$

$$\begin{tikzcd}
C_3 \arrow[d] & = & \mathbb{Z} \arrow[d, "0"] \\
C_2 \arrow[d] & = & \mathbb{Z} \arrow[d, "\times p"] \\
C_1 \arrow[d] & = & \mathbb{Z} \arrow[d, "0"] \\
C_0 & = & \mathbb{Z}
\end{tikzcd}
\quad \text{give} \quad
\begin{tikzcd}
C^3 & = \mathbb{Z}/p\mathbb{Z} \\
C^2 \arrow[u, "0"] & = \mathbb{Z}/p\mathbb{Z} \\
C^1 \arrow[u, "0"] & = \mathbb{Z}/p\mathbb{Z} \\
C^0 \arrow[u, "0"] & = \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

Hence, $H^k(L, \mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & k = 0, 1, 2, 3 \\ 0, & k > 3. \end{cases}$

- $p = 2$ corresponds to $\mathbb{R}P^3$.

Homology with coefficients: We take tensor products.

- Let A be an abelian group.
- Given a chain complex $C_*(X)$, we can consider the corresponding complex

$$C_*(X, A) = C_*(X) \otimes A.$$

- We get boundary maps $\partial_* \otimes id = \partial_*^A$ with $\partial_*^A \circ \partial_*^A = 0$
- The homology of $C_*(X, A)$ is $H_*(X, A)$

Universal coefficients thm: $\overset{= H_*(X, \mathbb{Z})}{H_*(X)}$ determines $H_*(X, A)$ for all abelian groups A

Example: We consider $A = \mathbb{Z}/p\mathbb{Z}$.

$$\begin{tikzcd}
C_3 \arrow[d] & = & \mathbb{Z} \arrow[d, "0"] \\
C_2 \arrow[d] & = & \mathbb{Z} \arrow[d, "\times p"] \\
C_1 \arrow[d] & = & \mathbb{Z} \arrow[d, "0"] \\
C_0 & = & \mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
C_3(L, \mathbb{Z}/p\mathbb{Z}) \arrow[d] & & & \mathbb{Z}/p\mathbb{Z} \arrow[d, "0"] \\
C_2(L, \mathbb{Z}/p\mathbb{Z}) \arrow[d] & & & \mathbb{Z}/p\mathbb{Z} \arrow[d, "0"] \\
C_1(L, \mathbb{Z}/p\mathbb{Z}) \arrow[d] & & & \mathbb{Z}/p\mathbb{Z} \arrow[d, "0"] \\
C_0(L, \mathbb{Z}/p\mathbb{Z}) & = \mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z} & = & \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

Hence $H_k(L, \mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & k = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$

Degree of a map:

$f: S^n \longrightarrow S^n$

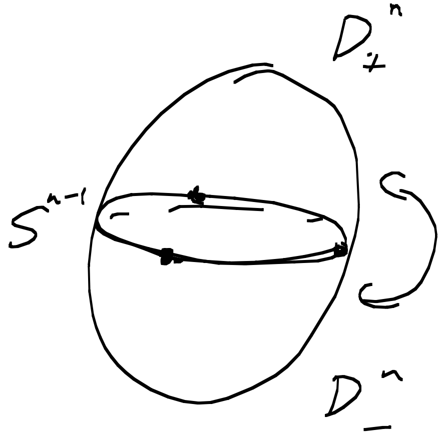
$f_*: H_n(S^n) \circlearrowleft$, $H_n(S^n) = \mathbb{Z}$

$\therefore$ $f_*(\xi) = k \cdot \xi$ for some $k \in \mathbb{Z}$ independent of ξ.

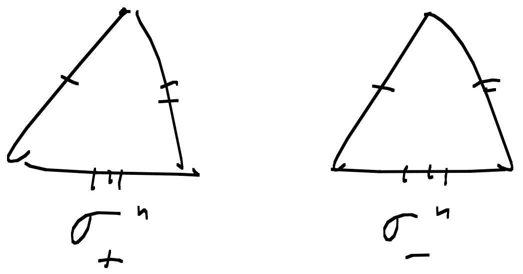
- k is called $\deg(f)$.

E.g. · $\deg(id) = 1$.

- If ρ is a reflection, $\deg(\rho) = -1$.



- S^n has a Δ-complex structure with 2 n-simplices $\sigma_\pm^n$ with faces identified using the identity.



- $C_2^\Delta(S^n) = \mathbb{Z}[\sigma_+^n, \sigma_-^n]$, $\partial_n \sigma_+^n = \partial_n \sigma_-^n = \sum_{\tau \text{ (n-1)-simplex}} \tau$

$\therefore H_n(S^n) = [\sigma_+^n - \sigma_-^n] = [\xi]$

- Reflection maps $\sigma_\pm^n$ to $\sigma_\mp^n$, so $\rho_\# : \xi \mapsto -\xi$.

$\therefore \rho_* : [\xi] \mapsto [-\xi]$.

Propn: $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Pf:

$$\begin{aligned} (f \circ g)_*(\xi) &= f_*(g_*(\xi)) \\ \overset{\shortparallel}{\deg(f \circ g)} \cdot \xi &= f_*(\deg(g) \cdot (\xi)) \\ &= \deg(g) \cdot f_*(\xi) \\ &= \deg(g) \cdot \deg(f)(\xi) \end{aligned}$$

$\square$.

Propn: The antipodal map $\tau : S^n \longrightarrow S^n$, $\tau : \tilde{x} \longmapsto -\tilde{x}$, has degree $(-1)^{n+1}$.

Pf: τ is a composition of $(n+1)$ reflections.

Cor: If n is even, then τ is not homotopic to the identity.

Ex: When n is odd, show that τ is homotopic to the identity.

Thm: There is no non-vanishing continuous vector field on S^n if n is even, i.e., we do not have $v : S^n \longrightarrow \mathbb{R}^{n+1}$ map s.t. $\langle v(x), x \rangle = 0 \;\; \forall \; x \in S^n$ & $\| v(x) \| = 1 \;\; \forall x$.

Pf: Define $H(x,t) = \cos(\pi t) \cdot x + \sin(\pi t) \cdot v(x)$, $x \in S^n$, $t \in [0,1]$.

$H : S^n \times [0,1] \longrightarrow S^n$, $H(x,0) = x$, $H(x,1) = -x$.

Thus, H is a homotopy between id & τ.

- But if n is even, $\deg(id) \neq \deg(\tau)$, contradicting Propn. below.

D.

<u>Propn:</u> $f \sim g \Rightarrow \deg(f) = \deg(g)$

Global & Local degree:

- Degree of $f(z) = z^n + a_1 z^{n-1} + \dots + a_0$.
- $\deg(f)$ is the number of solutions of $f(z) = c$ counted with multiplicity.
- If c is chosen appropriately, all roots are simple.

Key point: $\# f^{-1}(c)$ is independent of c as long as c avoids some bad values

namely,

- $f(z) = c$ has a multiple root

iff $\exists z_0$ s.t. $\begin{cases} f(z_0) = c \\ \& \quad f'(z_0) = 0 \end{cases}$

- We avoid $\underbrace{f(\mathcal{C})}_{\text{critical values.}}$, $\underbrace{\mathcal{C}}_{\text{critical points.}} = \{z \in \mathbb{C} : f'(z) = 0\}$

- For a complex polynomial, the set of critical values is finite.

- Sard's theorem: The set of critical values of a smooth function is of measure zero.

E.g. in real case: $f(x) = x^3 - x$

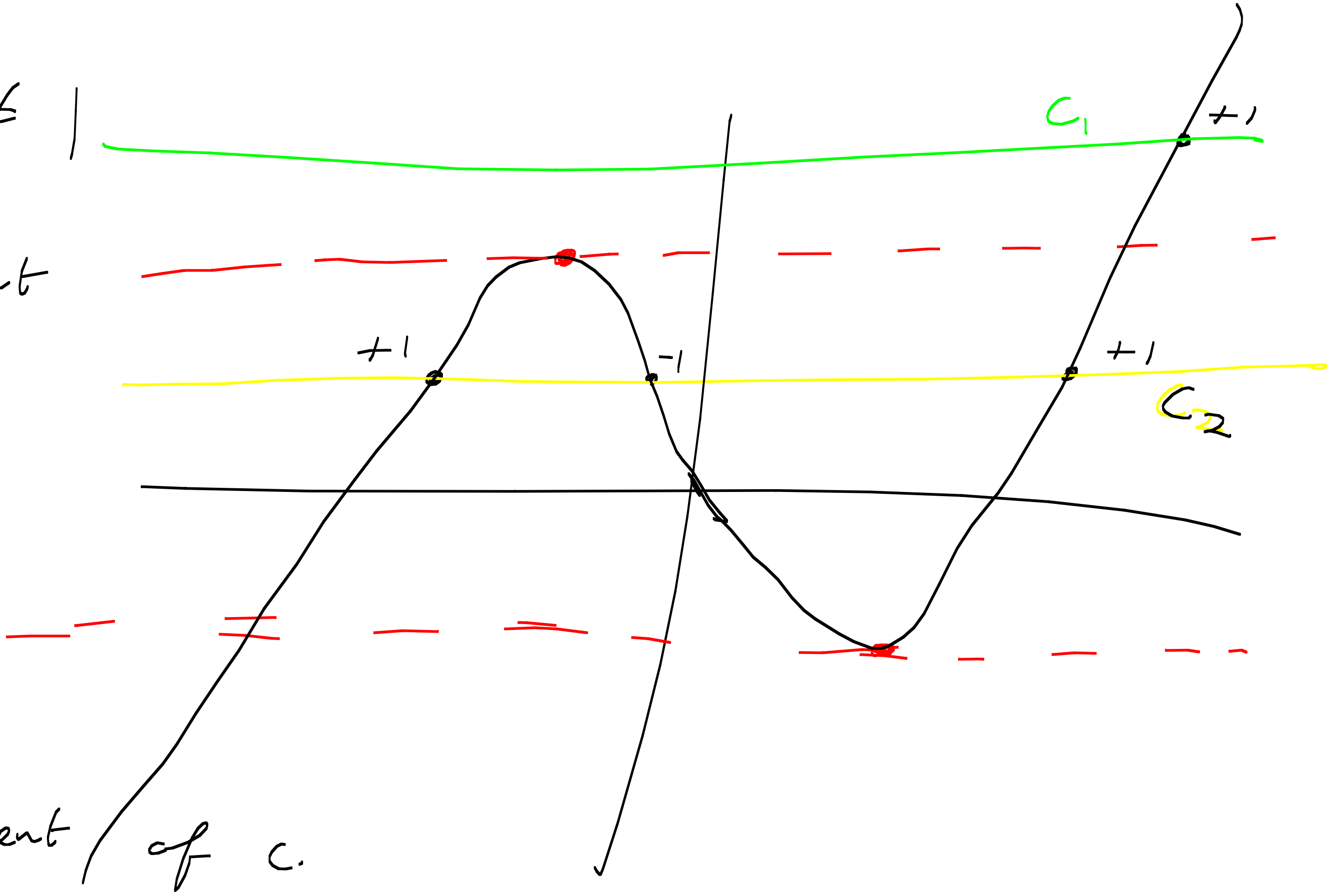
- $\# f^{-1}(c_1) = 1$
- $\# f^{-1}(c_2) = 3 \neq 1$

- Hence we count with sign

$\deg(f; c_1) = +1$

$\deg(f; c_2) = +1 - 1 + 1$
$= 1$

- This is independent of c.

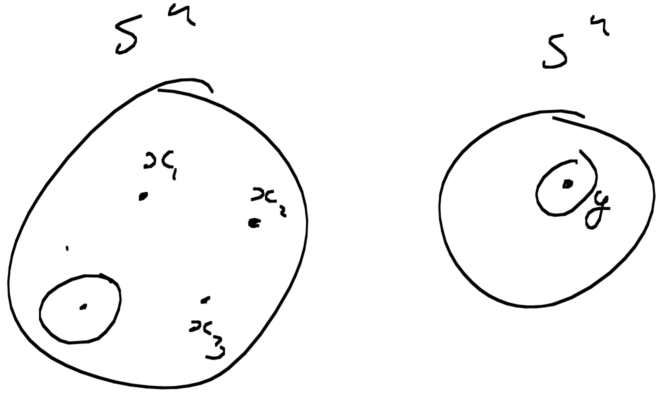


Degree & Local degree:

- $f : S^n \to S^n$, suppose $y \in S^n$ s.t. $f^{-1}(y)$ has no limit points $\Rightarrow$ $f^{-1}(y)$ is finite.

Rk: This is true if
- f smooth &
- $x \in f^{-1}(y) \Rightarrow Df(x)$ non-singular.

Let $f^{-1}(y) = \{x_1, \ldots, x_m\}$.



Local degree: Let $x_k \in f^{-1}(y)$,

- Let U_k be a nbd. of x_k disjoint from x_j, $j \neq k$.

Then $f : (U_k, U_k \setminus \{x_k\}) \to (S^n, S^n \setminus \{y\})$

Thus,

$$\begin{array}{ccc} f_*^{x_k} : H_n(U_k, U_k \setminus \{x_k\}) & \longrightarrow & H_n(S^n, S^n \setminus \{y\}) \\ \wr & & \wr \\ H_n(S^n, S^n \setminus \{x_k\}) & & H_n(S^n) \simeq \mathbb{Z} \\ \wr & & \\ H_n(S^n) \simeq \mathbb{Z} & & \end{array}$$

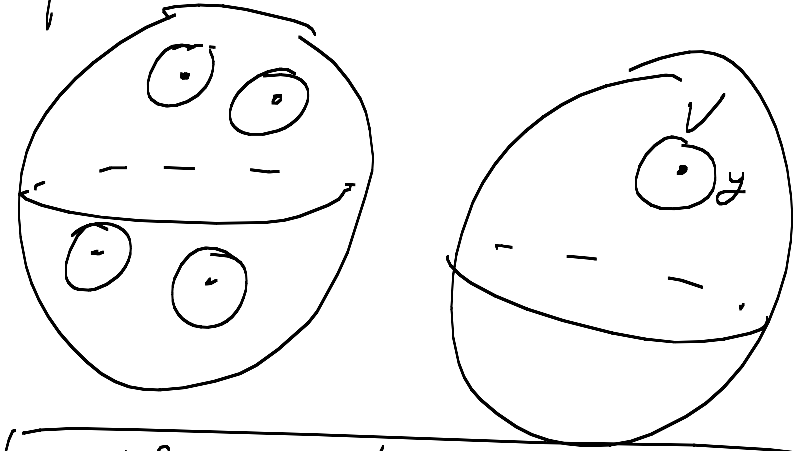
Thus, we have a homomorphism

$$f_*^{x_k} : \mathbb{Z} \to \mathbb{Z}$$

. The degree of this homomorphism is called the local degree, or. index $\text{ind}(f_* ; x_k)$.

Thm: $\deg(f) = \sum_{x_k \in f^{-1}(y)} \operatorname{ind}(f; x_k)$

V nbd. of y s.t. $f^{-1}(V) = \coprod_k U_k$, U_k nbd. of x_k.



Pf:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] & H_n(S^n) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[r] \arrow[d, equal] & H_n(S^n, S^n \setminus \{y\}) \\
H_n(\coprod U_k, \coprod(U_k \setminus x_k)) \arrow[ur] \arrow[d, equal] & \\
H_n(\coprod (U_k, U_k \setminus x_k)) \arrow[d, "\cong"] & \\
\bigoplus_k H_n(U_k, U_k \setminus \{x_k\}) \cong \bigoplus_k H_n(S^n) &
\end{tikzcd}$$

We get

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] \arrow[d, shift left=2, "{(1,\dots,1)}"] & H_n(S^n) \\
\bigoplus_k H_n(S^n) \arrow[ur] &
\end{tikzcd}$$

with k^{th} copy mapped by a map of degree $\operatorname{ind}(f; x_k)$.