Mockingbird lattices

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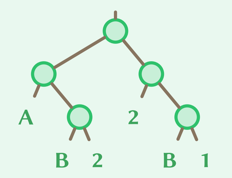
Combinatory logic and partial orders

Terms

- An alphabet is a finite set $\mathfrak{G}$ whose elements are basic combinators.
- A variable is any element of the set $\mathbb{X} := \{1, 2, \dots\}$.
- A $\mathfrak{G}$-term is a binary tree whose leaves are decorated on $\mathfrak{G} \cup \mathbb{X}$.
- Let $\mathfrak{T}(\mathfrak{G})$ be the set of the $\mathfrak{G}$-terms.
- A combinator is a $\mathfrak{G}$-term having no leaf decorated by a variable.

A $\mathfrak{G}$-term where $\mathfrak{G} = \{\mathbf{A}, \mathbf{B}\}$:



With the convention that each internal node ○ associates from left to right, this term is written as

$\mathbf{A}(\mathbf{B}\,2)(2(\mathbf{B}\,1))$.

Combinatory logic systems

- A rewrite relation is a binary relation $\to$ on $\mathfrak{T}(\mathfrak{G})$ such that
  $$\mathbf{X}\,1 \dots n \to \mathfrak{t}_{\mathbf{X}}$$
  where $\mathbf{X} \in \mathfrak{G}$ and $\mathfrak{t}_{\mathbf{X}}$ is a $\mathfrak{G}$-term where leaves are decorated on $[n]$.
- A combinatory logic system (CLS) is a pair $(\mathfrak{G}, \to)$ where $\mathfrak{G}$ is an alphabet and $\to$ is a rewrite relation.

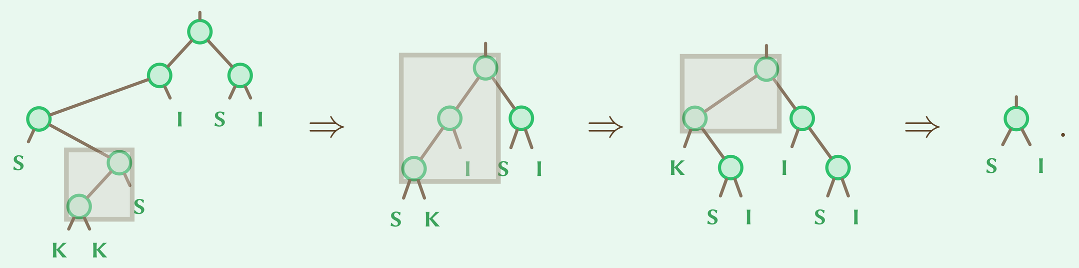
Here are some basic combinators with their rules:

- Identity bird: $\mathbf{I}1 \to 1$
- Mockingbird: $\mathbf{M}1 \to 1\,1$
- Kestrel: $\mathbf{K}1\,2 \to 1$
- Lark: $\mathbf{L}1\,2 \to 1(2\,2)$
- Cardinal: $\mathbf{C}1\,2\,3 \to 1\,3\,2$
- Bluebird: $\mathbf{B}1\,2\,3 \to 1(2\,3)$
- Starling: $\mathbf{S}1\,2\,3 \to 1\,3(2\,3)$
- Jay: $\mathbf{J}1\,2\,3\,4 \to 1\,2(1\,4\,3)$

A model of computation

- The closure of $\to$ is the binary relation $\Rightarrow$ on $\mathfrak{T}(\mathfrak{G})$ where $\mathfrak{t} \Rightarrow \mathfrak{t}'$ if
  (1) $\mathfrak{t} \to \mathfrak{t}'$;
  (2) or $\mathfrak{t} = \mathfrak{t}_1\,\mathfrak{t}_2$ and $\mathfrak{t}' = \mathfrak{t}_1'\,\mathfrak{t}_2$ with $\mathfrak{t}_1 \Rightarrow \mathfrak{t}_1'$;
  (3) or $\mathfrak{t} = \mathfrak{t}_1\,\mathfrak{t}_2$ and $\mathfrak{t}' = \mathfrak{t}_1\,\mathfrak{t}_2'$ with $\mathfrak{t}_2 \Rightarrow \mathfrak{t}_2'$.
- A $\mathfrak{G}$-term $\mathfrak{t}$ rewrites into a $\mathfrak{G}$-term $\mathfrak{t}'$ if $\mathfrak{t} \Rightarrow \mathfrak{t}'$.

For instance, if $\mathcal{C}$ is the CLS $(\mathfrak{G}, \to)$ where $\mathfrak{G} = \{\mathbf{I}, \mathbf{K}, \mathbf{S}\}$, we have

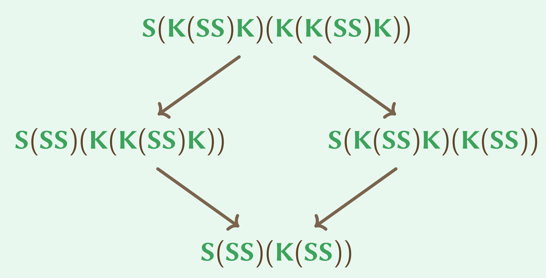


Rewrite graphs

Let $\mathcal{C}$ be a CLS.

- The reflexive and transitive closure of $\Rightarrow$ is the preorder $\preccurlyeq$.
- The symmetric closure of $\preccurlyeq$ is the equivalence relation $\equiv$.
- The rewrite graph $\mathrm{G}_{\mathcal{C}}$ of $\mathcal{C}$ is the digraph of the relation $\Rightarrow$ on $\mathfrak{T}(\mathfrak{G})$.
- $\mathcal{C}$ is locally finite if each $\equiv$-equivalence class is finite.

Here is a part of $\mathrm{G}_{\mathcal{C}}$ where $\mathcal{C}$ is the CLS of the previous example.

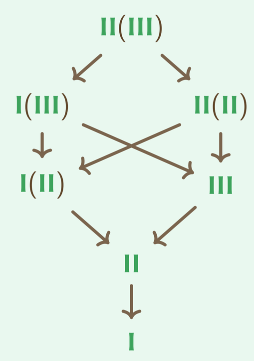


Order theoretic properties

Let $\mathcal{C}$ be a CLS.

- $\mathcal{C}$ has the poset property if $\preccurlyeq$ is a partial order relation.
  In this case the poset of $\mathcal{C}$ is the poset $\mathcal{P}_{\mathcal{C}} := (\mathfrak{T}(\mathfrak{G}), \preccurlyeq)$.
- $\mathcal{C}$ has the lattice property if $\mathcal{C}$ has the poset property and each interval of $\mathcal{P}_{\mathcal{C}}$ is a lattice.

Here is a part of $\mathrm{G}_{\mathcal{C}}$ where $\mathcal{C}$ is the CLS containing only $\mathbf{I}$.



This CLS has the poset property, but as shown by this Hasse diagram, it has not the lattice property.

A source of combinatorial questions

Main idea: *Use combinatory logic to construct new posets.*

Let $\mathcal{C}$ be a CLS.

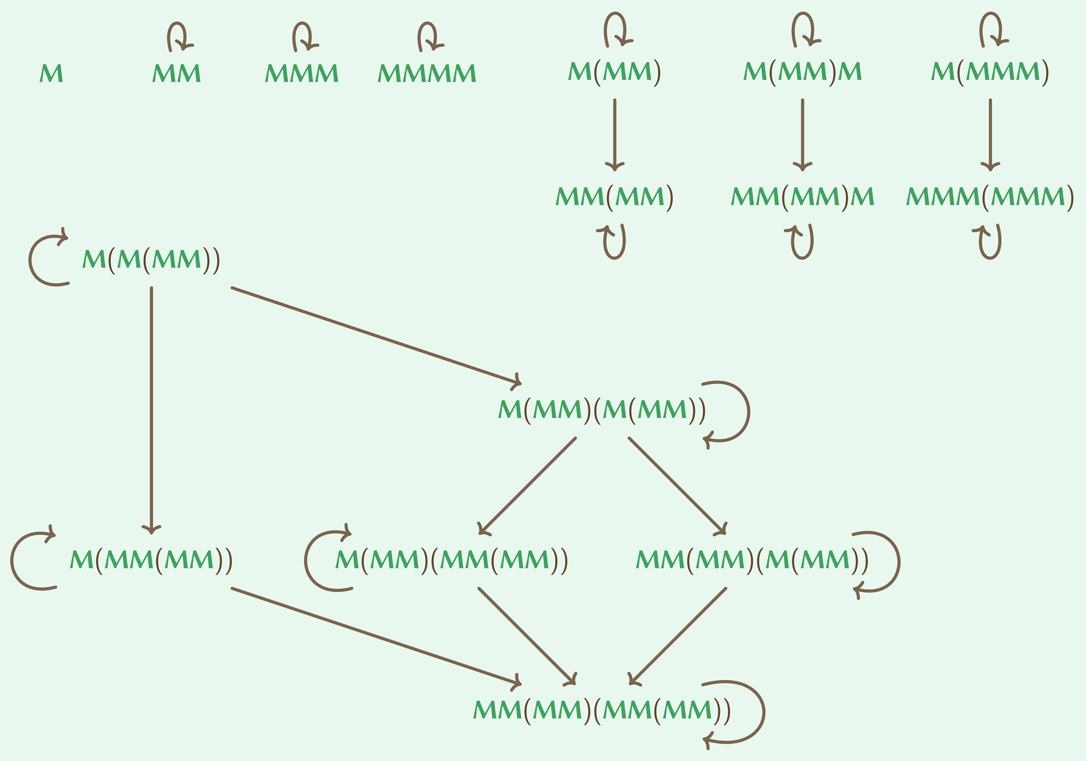
- Prove that $\mathcal{C}$ has the poset property.
  If it is the case, enumerate its minimal/maximal elements, its covering pairs, and its intervals.
- Prove that $\mathcal{C}$ has the lattice property.
  If it is the case, describe the meet and join operations of the lattices.

The Mockingbird lattices

The Mockingbird CLS

The Mockingbird CLS is the CLS $\mathcal{C} := (\mathfrak{G}, \to)$ where $\mathfrak{G} := \{\mathbf{M}\}$.

Here is a part of $\mathrm{G}_{\mathcal{C}}$:

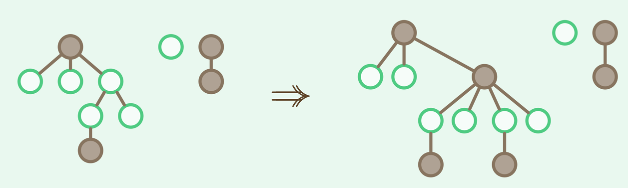


First properties of the Mockingbird CLS

- **Proposition.** *$\mathcal{C}$ is locally finite.*
- **Proposition.** *$\mathcal{C}$ has the poset property.*
- **Proposition.** *Each $\equiv$-equivalence class of $\mathcal{C}$ admits a least and a greatest element.*

Lattices of duplicative forests

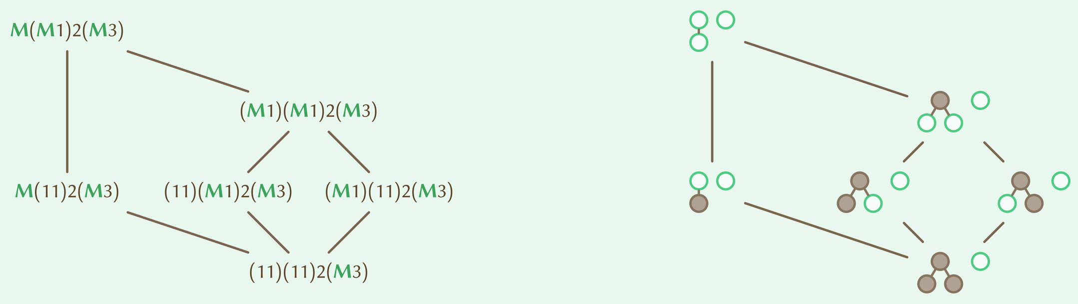
- A duplicative forest is a forest of planar rooted trees where nodes are either ● or ○.
- Let $\mathcal{D}^*$ be the set of the duplicative forests.
- Let $\Rrightarrow$ be the relation on $\mathcal{D}^*$ such that $\mathfrak{f} \Rrightarrow \mathfrak{f}'$ if $\mathfrak{f}'$ is obtained by blackening a white node of $\mathfrak{f}$ and by duplicating its sequence of descendants.
  For instance,



- Let $\ll$ be the reflexive and transitive closure of $\Rrightarrow$.
  Proposition. *The pair $(\mathcal{D}^*, \ll)$ is a poset.*
- There is more: **Proposition.** *$(\mathcal{D}^*, \ll)$ is a lattice.*

Lattice property

Theorem. *Any interval of $\mathcal{P}_{\mathcal{C}}$ is isomorphic to an interval of $(\mathcal{D}^*, \ll)$.*

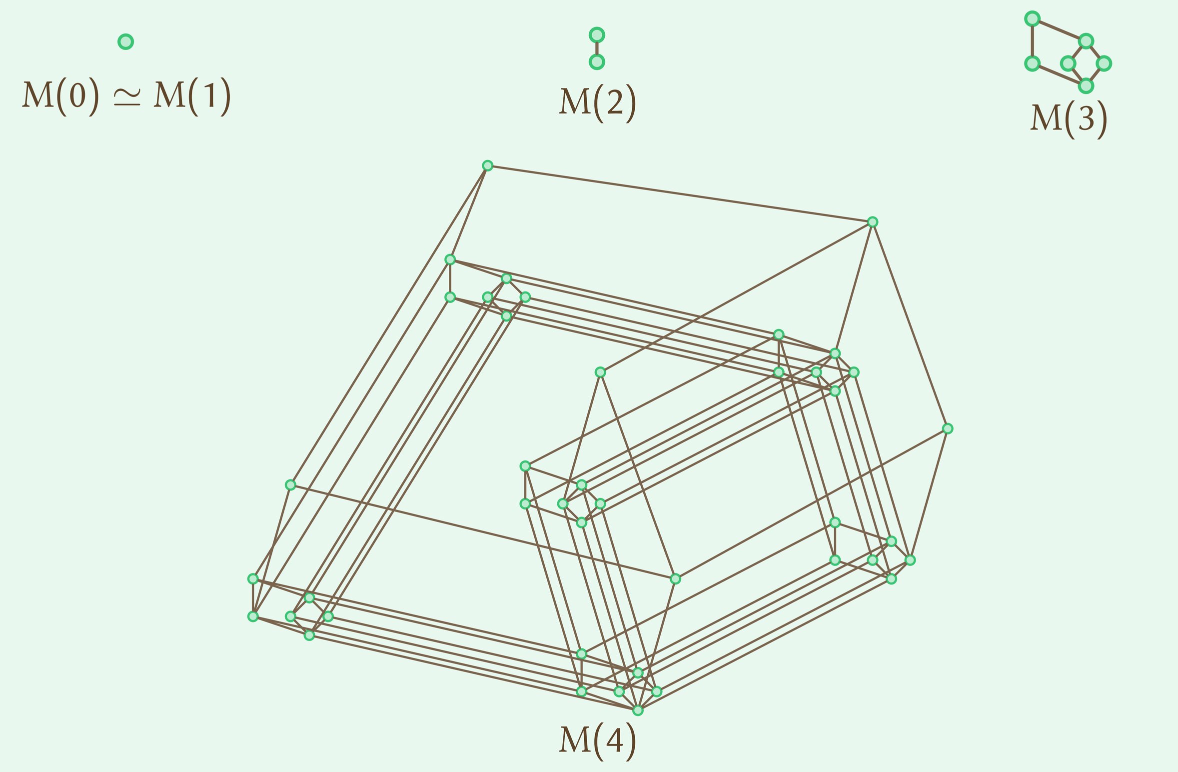


Therefore, $\mathcal{C}$ has the lattice property.

Mockingbird lattices

The Mockingbird lattice $\mathrm{M}(d)$ of order $d \geqslant 0$ is the upper set of $\mathcal{P}_{\mathcal{C}}$ generated by the right comb tree with $d+1$ leaves, all decorated by $\mathbf{M}$.

Here are the Hasse diagrams of the first Mockingbird lattices:



Some properties

Mockingbird lattices are

- not graded;
- not self-dual;
- not semi-distributive.

Enumerative properties

Tools

To establish the next results, we use the following tools.

- The lattice isomorphism between intervals of $\mathcal{P}_{\mathcal{C}}$ and of $\mathcal{D}^*$;
- The space $\mathbb{Q}\langle\langle\mathcal{D}^*\rangle\rangle$ of the formal power series of duplicative forests.
- To enumerate a set S of duplicative forests,
  - we express the characteristic series $\mathbf{F}_S$ of S;
  - we compute the image of $\mathbf{F}_S$ by the linear map sending $\mathfrak{f} \in \mathcal{D}^*$ to $z^{|\mathfrak{f}|}$.
- Two particular elements of $\mathbb{Q}\langle\langle\mathcal{D}^*\rangle\rangle$:
  - the series $\mathbf{Id}$ satisfying
    $$\mathbf{Id} = \epsilon + \circ + \text{(2-node white chain)} + \text{(3-node white chain)} + \cdots$$
  - the series $\mathbf{gr}_{\mathfrak{f}}$, $\mathfrak{f} \in \mathcal{D}^*$, satisfying
    $$\mathbf{gr}_{\mathfrak{f}} := \sum_{\mathfrak{f} \in \mathcal{D}^*} \sum_{\mathfrak{f} \ll \mathfrak{f}'} \mathfrak{f}'.$$
- The Hadamard product $\boxtimes$ on generating series.

Shortest and longest saturated chains in $\mathrm{M}(d)$

Proposition. *For any $d \geqslant 1$, in $\mathrm{M}(d)$, every*

- *shortest saturated chain has length d;*
- *longest saturated chain has length 2^{d-1}.*

Minimal and maximal elements in $\mathcal{P}_{\mathcal{C}}$

- **Proposition.** *The generating series $\mathrm{D}_{\min}$ of the closed minimal terms of $\mathcal{P}_{\mathcal{C}}$ enumerated w.r.t. their degrees satisfies*
  $$\mathrm{D}_{\min} = 1 + z + z\mathrm{D}_{\min}^2 - z\left(\mathrm{D}_{\min}\left[z := z^2\right]\right).$$
  The first numbers are
  1, 1, 2, 4, 12, 34, 108, 344, 1136, 3796, 12920.
- **Proposition.** *The generating series $\mathrm{D}_{\max}$ of the closed maximal terms of $\mathcal{P}_{\mathcal{C}}$ enumerated w.r.t. their degrees satisfies*
  $$\mathrm{D}_{\max} = 1 + z + z\mathrm{D}_{\max}^2 - z\mathrm{D}_{\max}.$$
  The first numbers are
  1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835,
  and form Sequence **A001006** (Motzkin numbers).
- **Proposition.** *The generating series $\mathrm{D}_{\mathrm{iso}}$ of the closed terms of $\mathcal{P}_{\mathcal{C}}$ that are both minimal and maximal enumerated w.r.t. their degrees satisfies*
  $$\mathrm{D}_{\mathrm{iso}} = 1 + 2z + z\mathrm{D}_{\mathrm{iso}}^2 - z\mathrm{D}_{\mathrm{iso}} - z\left(\mathrm{D}_{\mathrm{iso}}\left[z := z^2\right]\right).$$
  The first numbers are
  1, 1, 1, 1, 3, 5, 13, 29, 71, 171, 427.

Elements, covering pairs and intervals in $\mathrm{M}(d)$

- **Theorem.** *The generating series H_{gr} of the elements of $\mathrm{M}(d)$ enumerated w.r.t. $d \geqslant 0$ satisfies*
  $$\mathrm{H}_{\mathrm{gr}} = 1 + z\mathrm{H}_{\mathrm{gr}} + z\left(\mathrm{H}_{\mathrm{gr}} \boxtimes \mathrm{H}_{\mathrm{gr}}\right).$$
  The first numbers are
  1, 1, 2, 6, 42, 1806, 3263442, 10650056950806,
  and form Sequence **A007018**.
- **Theorem.** *The generating series H_{ni} of the covering pairs of $\mathrm{M}(d)$ enumerated w.r.t. $d \geqslant 0$ satisfies*
  $$\mathrm{H}_{\mathrm{ni}} = z\mathrm{H}_{\mathrm{ni}} + z\mathrm{H}_{\mathrm{gr}} + 2z\left(\mathrm{H}_{\mathrm{ni}} \boxtimes \mathrm{H}_{\mathrm{gr}}\right).$$
  The first numbers are
  0, 0, 1, 7, 97, 8287, 29942737, 195432804247687.
- **Theorem.** *The generating series $\mathrm{H}_{\mathrm{ns}} = \mathrm{H}_{\mathrm{ns}}^{(1)}$ of the intervals of $\mathrm{M}(d)$ enumerated w.r.t. $d \geqslant 0$ satisfies $\mathrm{H}_{\mathrm{ns}} = \mathrm{H}_{\mathrm{ns}}^{(1)}$ where, for any $k \geqslant 1$, $\mathrm{H}_{\mathrm{ns}}^{(k)}$ satisfies*
  $$\mathrm{H}_{\mathrm{ns}}^{(k)} = 1 + z\left(\mathrm{H}_{\mathrm{ns}}^{(k)} \boxtimes \mathrm{H}_{\mathrm{ns}}^{(k)}\right) + z\sum_{i \in [\![k]\!]} \binom{k}{i} \mathrm{H}_{\mathrm{ns}}^{(k+i)}.$$
  The first numbers are
  1, 1, 3, 17, 371, 144513, 20932611523, 438176621806663544657.

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