Equidistributions around special kinds of descents and excedances via continued fractions

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Introduction

We consider a sequence of four variable polynomials by refining Stieltjes' continued fraction for Eulerian polynomials. Using the combinatorial theory of Jacobi-type continued fractions and bijections we derive various combinatorial interpretations in terms of permutation statistics for these polynomials, which include special kinds of descents and excedances in a recent paper of Baril and Kirgizov. As a by-product, we derive several equidistribution results for permutation statistics, which enables us to confirm and strengthen a recent conjecture of Vajnovszki and also to obtain several companion permutation statistics for two bistatistics in a conjecture of Baril and Kirgizov.

Definition 1. *Eulerian polynomials* $A_n(t) := \sum_{k\geq 0} A_{n,k}t^k$ *by*

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{(1-t)e^z}{e^{zt}-te^z}. \qquad (1)$$

Definition 2. *Let* $\mathfrak{S}_n$ *is the set of permutations on* $\{1,\ldots,n\}$.

$$\operatorname{des}\sigma = \#\{i\in[n-1] | \sigma(i)>\sigma(i+1)\},$$
$$\operatorname{exc}\sigma = \#\{i\in[n] | \sigma(i)>i\},$$

Proposition 1 (Riordan1958, MacMahon1913)**.**

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}$$

For a permutation $\sigma := \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1\ldots n$, an index $i\in[1,\, n-1]$ is called a

- *descent* (resp. *excedance*) if $\sigma(i)>\sigma(i+1)$ (resp. $\sigma(i)>i$);
- *descent of type 2* if i is a descent and $\sigma(j)<\sigma(i)$ for $j<i$;
- *pure excedance* if i is an excedance and $\sigma(j)\notin[i,\sigma(i)]$ for $j<i$;

and an index $i\in[2,\, n]$ is called a

- *drop* if $i>\sigma(i)$;
- *pure drop* if i is a drop and $\sigma(j)\notin[\sigma(i),i]$ for $j>i$.

Let $\operatorname{des}\sigma$ (resp. $\operatorname{exc}\sigma$, $\operatorname{drop}\sigma$, $\operatorname{des}_2\sigma$, $\operatorname{pex}\sigma$ and $\operatorname{pdrop}\sigma$) denote the number of descents (resp. excedances, drops, descents of type 2, pure excedances and pure drops) of σ.

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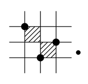
Mesh patterns where first introduced by Brändén and Claesson (2011), as a further extension of bivincular patterns. A pair (τ,R), where τ is a permutation in S_k and R is a subset of $[\![0,k]\!]\times[\![0,k]\!]$, where $[\![0,k]\!]$ denotes the interval of the integers from 0 to k, is a *mesh pattern* of length k.

Let (i,j) denote the box whose corners have coordinates $(i,j),(i,j+1),(i+1,j+1)$ and $(i+1,j)$. An example of a mesh pattern is the classical pattern 312 along with $R=\{(1,2),(2,1)\}$. We draw this by shading the boxes in R

.

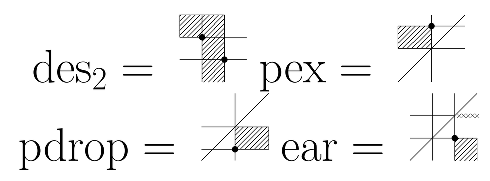


Figure 1: Illustration of the mesh patterns des_2 and pex and ear, where the cross line means that the value cannot be in the segment of the horizontal line

Recently Baril and Kirgizov proved the equidistribution of the statistics "des_2", "pex" and "pcyc" over $\mathfrak{S}_n$ by bijections and conclude their paper with the following two conjectures on the equidistribution of two pairs of bistatistics.

Conjecture 1 (Baril and Kirgizov)**.** *The two bistatistics* $(\operatorname{des}_2,\operatorname{cyc})$ *and* $(\operatorname{pex},\operatorname{cyc})$ *are equidistributed on* $\mathfrak{S}_n$.

Conjecture 2 (Vajnovszki)**.** *The two bistatistics* $(\operatorname{des}_2,\operatorname{des})$ *and* $(\operatorname{pex},\operatorname{exc})$ *are equidistributed on* $\mathfrak{S}_n$.

Refined Eulerian polynomials by continued fractions

In this paper we shall take a different approach to their problems through the combinatorial theory of J-continued fractions developed by Flajolet and Viennot in the 1980's. Recall that a J-type continued fraction is a formal power series defined by

$$\sum_{n=0}^{\infty} a_n z^n = \cfrac{1}{1-\gamma_0 z - \cfrac{\beta_1 z^2}{1-\gamma_1 z - \cfrac{\beta_2 z^2}{\cdots}}},$$

where $(\gamma_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ are two sequences in some commutative ring.

Define the polynomials $A_n(t,\lambda,y,w)$ by the J-fraction

$$\sum_{n\geq 0} z^n A_n(t,\lambda,y,w) = \cfrac{1}{1-wz-\cfrac{t\lambda y\, z^2}{1-(w+t+1)z-\cfrac{t(\lambda+1)(y+1)\, z^2}{\cdots}}} \qquad (2)$$

with $\gamma_n = w+n(t+1)$ and $\beta_n = t(\lambda+n-1)(y+n-1)$.

For $\sigma\in\mathfrak{S}_n$, an index $i\in[n]$ is called a

- *cycle peak* (cpeak) if $\sigma^{-1}(i)<i>\sigma(i)$;
- *cycle valley*/ (cval) if $\sigma^{-1}(i)>i<\sigma(i)$;
- *cycle double rise* (cdrise) if $\sigma^{-1}(i)<i<\sigma(i)$;
- *cycle double fall* (cdfall) if $\sigma^{-1}(i)>i>\sigma(i)$;
- *fixed point* (fix) if $\sigma^{-1}(i)=i=\sigma(i)$.

Clearly every index i belongs to exactly one of these five types; we refer to this classification as the *cycle classification*. Next, an index $i\in[n]$ (or a value $\sigma(i)$) is called a

- *record* (rec) (or *left-to-right maximum*) if $\sigma(j)<\sigma(i)$ for all $j<i$ (the index 1 is always a record];
- *antirecord* (arec) (or *right-to-left minimum*) if $\sigma(j)>\sigma(i)$ for all $j>i$ (the index n is always an antirecord);
- *exclusive record* (erec) if it is a record and not also an antirecord;
- *exclusive antirecord* (earec) if it is an antirecord and not also a record.
- *exclusive antirecord cycle peak* (ear) if i is an exclusive antirecord and also a cycle peak.

Main results

Theorem 2. *We have*

$$\begin{aligned}A_n(t,\lambda,y,w) &= \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}\lambda^{\operatorname{pex}\sigma}y^{\operatorname{ear}\sigma}w^{\operatorname{fix}\sigma}\\ &= \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}\lambda^{\operatorname{pcyc}\sigma}y^{\operatorname{ear}\sigma}w^{\operatorname{fix}\sigma}\\ &= \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}\lambda^{\operatorname{pcyc}\sigma}y^{\operatorname{pex}\sigma}w^{\operatorname{fix}\sigma}.\end{aligned}$$

By (2), the polynomial $A_n(t,\lambda,y,w)$ is invariant under $\lambda\leftrightarrow y$. Hence, the above theorem implies immediately the following result.

Corollary 3. *The six bistatistics* $(\operatorname{pex},\operatorname{ear})$, $(\operatorname{ear},\operatorname{pex})$, $(\operatorname{ear},\operatorname{pcyc})$, $(\operatorname{pcyc},\operatorname{ear})$, $(\operatorname{pex},\operatorname{pcyc})$ *and* $(\operatorname{pcyc},\operatorname{pex})$ *are equidistributed on* $\mathfrak{S}_n$.

Now we consider three specializations of $A_n(t,\lambda,y,w)$. First let $B_n(t,\lambda,w)=A_n(t,\lambda,1,w)=A_n(t,1,\lambda,w)$, namely,

$$\sum_{n\geq 0} z^n B_n(t,\lambda,w) = \cfrac{1}{1-wz-\cfrac{t\lambda\, z^2}{1-(w+t+1)z-\cfrac{2t(\lambda+1)\, z^2}{\cdots}}} \qquad (4)$$

with $\gamma_n = w+n(t+1)$ and $\beta_n = nt(\lambda+n-1)$.

To deal with descent statistics, we recall some linear statistics. For $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)\in\mathfrak{S}_n$ with convention 0–∞, i.e., $\sigma(0)=0$ and $\sigma(n+1)=n+1$, a value $\sigma(i)$ $(1\leq i\leq n)$ is called a

- *double ascent* (dasc) if $\sigma(i-1)<\sigma(i)$ and $\sigma(i)<\sigma(i+1)$;
- *double descent* (ddes) if $\sigma(i-1)>\sigma(i)$ and $\sigma(i)>\sigma(i+1)$;
- *peak* (peak) if $\sigma(i-1)<\sigma(i)$ and $\sigma(i)>\sigma(i+1)$;
- *valley* (valley) if $\sigma(i-1)>\sigma(i)$ and $\sigma(i)<\sigma(i+1)$.

A double ascent $\sigma(i)$ $(1\leq i\leq n)$ is called a *foremaximum* of σ if it is at the same time a record. Denote the number of foremaxima of σ by $\operatorname{fmax}\ \sigma$. For example, if $\sigma = 3\,4\,2\,1\,5\,8\,7\,6$, then $\operatorname{dasc}\sigma=\operatorname{ddes}\sigma=\operatorname{peak}\sigma=\operatorname{val}\sigma=2$ and $\operatorname{fmax}\ \sigma=2$ as the foremaxima of σ are $3,5$.

Theorem 4. *We have*

$$\begin{aligned}B_n(t,\lambda,w) &= \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}\lambda^{\operatorname{pcyc}\sigma}w^{\operatorname{fix}\sigma} && (5a)\\ &= \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}\lambda^{\operatorname{ear}\sigma}w^{\operatorname{fix}\sigma} && (5b)\\ &= \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}\lambda^{\operatorname{pex}\sigma}w^{\operatorname{fix}\sigma} && (5c)\\ &= \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma}\lambda^{\operatorname{des}_2\sigma}w^{\operatorname{fmax}\sigma} && (5d)\end{aligned}$$

and

$$\sum_{n\geq 0} B_n(t,\lambda,w)\frac{z^n}{n!} = e^{wz}\left(\frac{1-t}{e^{tz}-te^z}\right)^{\lambda}. \qquad (5e)$$

B. Han, J. Mao and J. Zeng, Equidistributions around special kinds of descents and excedances. SIAM J. Discrete Math. 35 (2021), no. 4, 2858–2879.