Factorization of classical characters twisted by roots of unity

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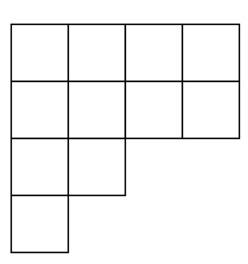
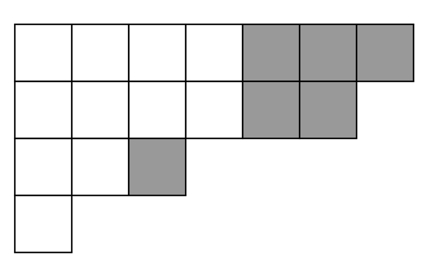
Abstract

- Fix $t \geq 2$, $n \in \mathbb{Z}^+$. Consider the irreducible characters of representations of GL_{tn}, SO_{2tn+1}, Sp_{2tn} and O_{2tn} over $\mathbb{C}$, evaluated at elements $\omega^k x_i$ for $0 \leq k \leq t-1$ and $1 \leq i \leq n$, where ω is a primitive t^{th} root of unity.
- Motivated by the case of GL_{tn}, considered by D. J. Littlewood (AMS press, 1950) and independently by D. Prasad (Israel J. Math., 2016).
- We characterize partitions for which the specialized irreducible character is nonzero in terms of what we call z-asymmetric partitions, where z is an integer which depends on the group.
- The non-zero character factorizes into characters of smaller classical groups.
- We also give product formulas for general z-asymmetric partitions and t-cores.
- Finally, we show that there are infinitely many z-asymmetric t-cores for $t \geq z+2$.

Notations and Definitions

- $X = (x_1, \dots, x_n)$ - a tuple of commuting indeterminates. $X^j = (x_1^j, \dots, x_n^j)$, $j \in \mathbb{Z}$. $\overline{X} = \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right)$.
- Partition and its beta-set:



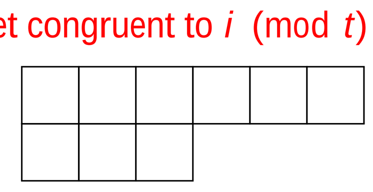


$\lambda = (4,4,2,1) \vdash 11$; $\ell(\lambda) = 4$; $\text{rk}(\lambda) = 2$ $\qquad \beta(\lambda;4) = (7,6,3,1)$

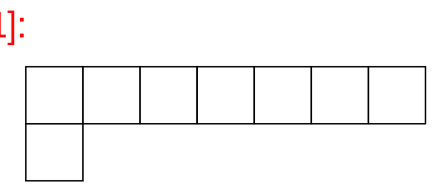
- For $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_j)$ partitions, $k + j \leq 2n$,

$$\lambda_1 + (\lambda, 0, \dots, 0, -\text{rev}(\mu)) = (\lambda_1 + \lambda_1, \dots, \lambda_1 + \lambda_k, \underbrace{\lambda_1, \dots, \lambda_1}_{2n-j-k}, \lambda_1 - \mu_j, \dots, \lambda_1 - \mu_1, 0).$$

- The parts of the beta set congruent to $i \pmod t$ for $i \in [0, t-1]$:

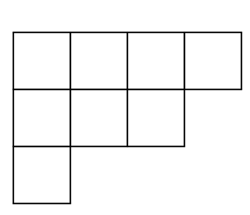


$i = 0$; $t = 3$



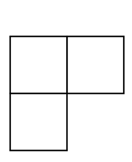
$i = 1$; $t = 3$

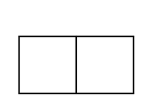
- t-core of λ: Consider $tj + i$; $0 \leq j \leq n_i(\lambda; m) - 1$; $0 \leq i \leq t-1$



$\text{core}_3(\lambda) = (4-3, 3-2, 1-1, 0) = (1,1)$.

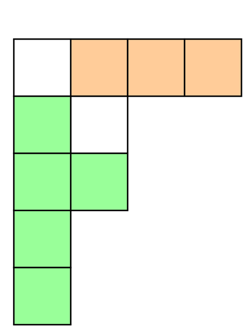
- t-quotient of λ: For each $i \in [0, t-1]$, consider $\frac{\beta^{(i)}(\lambda; m)}{t}$





$\text{quo}_3(\lambda) = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)})$, $\lambda^{(0)} = (2-1, 1-0) = (1,1)$, $\lambda^{(1)} = (2-1, 0-0) = (1)$, $\lambda^{(2)} = \emptyset$.

- z-asymmetric partition: $(\alpha \mid \alpha + z)$, $z \in \mathbb{Z}$.



$\lambda = (4,2,2,1,1) = (3,0 \mid 4,1)$
symplectic (1-asymmetric) 3-core

Weyl Character Formulas

Let λ be a partition of length at most n.

- The Schur polynomial or general linear (type A) character of GL_n indexed by λ:

$$s_\lambda(X) = \frac{\det_{1 \leq i,j \leq n}\left(x_i^{\beta_j(\lambda;n)}\right)}{\det_{1 \leq i,j \leq n}\left(x_i^{n-j}\right)}.$$

- The odd orthogonal (type B) character of the group $SO(2n+1)$ indexed by λ:

$$so_\lambda(X) = \frac{\det_{1 \leq i,j \leq n}\left(x_i^{\beta_j(\lambda;n)+1/2} - x_i^{-\beta_j(\lambda;n)-1/2}\right)}{\det_{1 \leq i,j \leq n}\left(x_i^{n-j+1/2} - x_i^{-(n-j+1/2)}\right)}.$$

- The symplectic (type C) character of the group $Sp(2n)$ indexed by λ:

$$sp_\lambda(X) = \frac{\det_{1 \leq i,j \leq n}\left(x_i^{\beta_j(\lambda;n)+1} - x_i^{-\beta_j(\lambda;n)-1}\right)}{\det_{1 \leq i,j \leq n}\left(x_i^{n-j+1} - x_i^{-(n-j+1)}\right)}.$$

- The even orthogonal (type D) character of the group $O(2n)$ indexed by λ:

$$o_\lambda^{\text{even}}(X) = \frac{2 \det_{1 \leq i,j \leq n}\left(x_i^{\beta_j(\lambda;n)} + x_i^{-\beta_j(\lambda;n)}\right)}{(1 + \delta_{\lambda_n,0}) \det_{1 \leq i,j \leq n}\left(x_i^{n-j} + x_i^{-(n-j)}\right)},$$

where δ is the Kronecker delta.

- For $\ell(\lambda) \leq tn$, let $\sigma_\lambda \in S_{tn}$ be the permutation that rearranges the parts of $\beta(\lambda; tn)$ such that

$$\beta_{\sigma_\lambda(j)}(\lambda; tn) \equiv q \pmod t; \quad \sum_{i=0}^{q-1} n_i(\lambda; tn) + 1 \leq j \leq \sum_{i=0}^{q} n_i(\lambda; tn),$$

arranged in decreasing order for each $q \in \{0, 1, \dots, t-1\}$.

Schur Factorization

Theorem (D. J. Littlewood (AMS press, 1950), D. Prasad (*Israel J. Math., 2016*))

Let λ be a partition of length at most tn indexing an irreducible representation of GL_{tn} and $\text{quo}_t(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(t-1)})$. Then the Schur polynomial $s_\lambda(X, \omega X, \dots, \omega^{t-1} X)$ is given as follows.

1. If $\text{core}_t(\lambda)$ is non-empty, then
$$s_\lambda(X, \omega X, \dots, \omega^{t-1} X) = 0.$$
2. If $\text{core}_t(\lambda)$ is empty, then
$$s_\lambda(X, \omega X, \dots, \omega^{t-1} X) = \text{sgn}(\sigma_\lambda)(-1)^{\frac{n(n+1)}{2}\frac{t(t-1)}{2}} \prod_{i=0}^{t-1} s_{\lambda^{(i)}}(X^t).$$

Factorization of other Classical Characters

Theorem (Ayyer-Kumari, [1], 2021)

Let λ be a partition of length at most tn indexing an irreducible representation of Sp_{2tn} and $\text{quo}_t(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(t-1)})$. Then the Sp_{2tn}-character $sp_\lambda(X, \omega X, \dots, \omega^{t-1} X)$ is given as follows.

1. If $\text{core}_t(\lambda)$ is not a symplectic t-core, then
$$sp_\lambda(X, \omega X, \dots, \omega^{t-1} X) = 0.$$
2. If $\text{core}_t(\lambda)$ is a symplectic t-core with rank r, then
$$sp_\lambda(X, \omega X, \dots, \omega^{t-1} X) = (-1)^{\epsilon} \text{sgn}(\sigma_\lambda) sp_{\lambda^{(t-1)}}(X^t) \prod_{i=0}^{\lfloor \frac{t-3}{2} \rfloor} s_{\mu_i}(X^t, \overline{X}^t) \begin{cases} so_{\lambda^{(\frac{t}{2}-1)}}(X^t) & t \text{ even}, \\ 1 & t \text{ odd}, \end{cases}$$
where
$$\epsilon = \sum_{i=\lfloor \frac{t}{2} \rfloor}^{t-2} \binom{n_i(\lambda)+1}{2} + \begin{cases} \frac{n(n+1)}{2} + nr & t \text{ even}, \\ 0 & t \text{ odd}; \end{cases}$$
and $\mu_i = \lambda_1^{(t-2-i)} + (\lambda^{(i)}, 0, \dots, 0, -\text{rev}(\lambda^{(t-2-i)}))$, $0 \leq i \leq \lfloor \frac{t-3}{2} \rfloor$.

- Example: $t = 2$, $n = 1$ and $a \geq b \geq 0$. $sp_{(a,b)}(x, -x)$ is nonzero if and only if a and b have the same parity.

$$sp_{(a,b)}(x, -x) = \begin{cases} -sp_{(\frac{b-1}{2})}(x^2)\, so_{(\frac{a+1}{2})}(x^2) & a \text{ and } b \text{ are odd}, \\ sp_{(\frac{a}{2})}(x^2)\, so_{(\frac{b}{2})}(x^2) & a \text{ and } b \text{ are even}. \end{cases}$$

- We give similar factorization results for the irreducible characters of classical groups of type B and D, namely O_{2tn} [1, Theorem 2.15] and SO_{2tn+1} [1, Theorem 2.17], where we specialize the elements as before.

Generating Functions

- The set of z-asymmetric partitions and z-asymmetric t-cores - P_z and $P_{z;t}$ respectively.

Theorem (Ayyer-Kumari, [1], 2021)

For $z \in \mathbb{Z}$,
$$\sum_{\lambda \in P_z} q^{|\lambda|} = \prod_{k \geq 0}(1 + q^{z+1+2k}) = (-q^{z+1}; q^2)_\infty, \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

Theorem (Ayyer-Kumari, [1], 2021)

For $|z| \geq t-1$, the empty partition is the only t-core in $P_{z;t}$.

Theorem (Ayyer-Kumari, [1], 2021)

Let $0 \leq z \leq t-2$. Represent elements of $\mathbb{Z}^{\lfloor \frac{t-z}{2} \rfloor}$ by $(z_0, \dots, z_{\lfloor \frac{t-z-2}{2} \rfloor})$ and define $b \in \mathbb{Z}^{\lfloor \frac{t-z}{2} \rfloor}$ by $b_i = t - z - 1 - 2i$.

Then there exists a bijection $\phi: P_{z;t} \to \mathbb{Z}^{\lfloor \frac{t-z}{2} \rfloor}$ satisfying $|\lambda| = t\|\phi(\lambda)\|^2 - b \cdot \phi(\lambda)$, where $\cdot$ represents the standard inner product.

- Ramanujan theta function:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}.$$

Corollary (Ayyer-Kumari, [1], 2021)

Let $p_{z;t}(m)$ be the cardinality of partitions in $P_{z;t}$ of size m. For $0 \leq z \leq t-2$, we have

$$\sum_{m \geq 0} p_{z;t}(m) q^m = \prod_{i=0}^{\lfloor (t-z-2)/2 \rfloor} f(q^{2i+z+1}, q^{2t-2i-z-1}).$$

Reference

[1] A. Ayyer, N. Kumari, *Factorization of Classical characters twisted by roots of unity*, to appear in Journal of Algebra. arXiv identifier: 2109.11310.