Harmonic Polynomials on Perfect Matchings

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Abstract

- We show that functions over perfect matchings of complete graphs admit unique (canonical) presentations as harmonic polynomials annihilated by certain differential operators (**Theorem 1**).
- Using the theory of Jack polynomials, we give a concrete description of these harmonic polynomials by computing the unique harmonic presentation of the standard basis of Specht polynomials (**Theorem 2**).
- We prove a perhaps new combinatorial identity that equates the product of the top row of lower hook lengths of λ to a weighted sum of so-called *tableau transversals* of λ (**Theorem 3**).

In this poster presentation, we focus just on perfect matchings of the complete bipartite graph $K_{n,n}$, that is, *the symmetric group* S_n.

Polynomial Presentations of Functions

Let $f \in \mathbb{R}S_n$ be a real-valued function on the symmetric group. Let $p \in \mathbb{R}[X]$ be a polynomial in the variables

$$X = \begin{pmatrix} X_{1,1}, \cdots, X_{1,n} \\ \vdots, \cdots, \vdots \\ X_{n,1}, \cdots, X_{n,n} \end{pmatrix}.$$

Let $P_\sigma \in GL_n$ be the permutation matrix of $\sigma \in S_n$.
We say $p \in \mathbb{R}[X]$ is a *presentation* of $f \in \mathbb{R}S_n$ if

$$f(\sigma) = p(P_\sigma) \quad \text{for all } \sigma \in S_n,$$

and we write $f \equiv p$.

Note that $X_{i,j}X_{i,k} \equiv 0$ and $X_{i,k}X_{j,k} \equiv 0$ for all $1 \le i,j,k \le n$.
A polynomial p is *succinct* if its monomial terms are not multiples of $X_{i,j}X_{i,k}$ or $X_{i,k}X_{j,k}$ for all $1 \le i,j,k \le n$.

We can still present $0 \in \mathbb{R}S_n$ as a succinct polynomial z:

$$z(X) = \sum_{i=1}^n \sum_{j=1}^n (l_i + r_j) X_{i,j} \ \text{ such that } \ \sum_{i=1}^n l_i + \sum_{j=1}^n r_j = 0,$$

thus there is no unique succinct presentation of any $f \in \mathbb{R}S_n$.

Is there a canonical succinct presentation of each $f \in \mathbb{R}S_n$?

Yes, if we further insist the polynomial is harmonic.

Harmonic Polynomials

We say that a succinct polynomial $p \in \mathbb{R}[X]$ is *harmonic* if

$$\Delta_{i,*}p := \sum_{j=1}^n \partial p/\partial X_{i,j} = 0 \quad \forall 1 \le i \le n, \text{ and}$$

$$\Delta_{*,j}p := \sum_{i=1}^n \partial p/\partial X_{i,j} = 0 \quad \forall 1 \le j \le n.$$

Theorem 1 *Any $f \in \mathbb{R}S_n$ can be presented uniquely as a succinct harmonic polynomial $p \in \mathbb{R}[X]$. Moreover, the unique succinct harmonic presentation of the $\perp$-projection $f^{=d}$ of f onto V_d (see below) equals the dth homogeneous part $p^{=d}$ of p.*

$$\mathbb{R}S_n \cong \bigoplus_{d=0}^{n-1} V^d, \quad V^d := \bigoplus_{\lambda \vdash n : \lambda_1 = n-d} V^\lambda$$

The proof features a class of incidence matrices that we call the *matching inclusion matrices* $W_{\ell,n}$ whose rows are indexed by partial matchings of size ℓ, columns indexed by perfect matchings, defined such that

$$W_{\ell,n}[m,M] = \begin{cases} 1 & \text{if } m \subseteq M; \\ 0 & \text{otherwise.} \end{cases}$$

- The matching-analogue of the celebrated *set incidence matrices*.
- Experimental data shows the nonzero elementary divisors of $W_{\ell,n}$ for all $\ell \le n \le 6$ are 1. *Is this true for all n?*

Jack Polynomials

For any $\alpha \in \mathbb{R}$, the *(integral form) Jack polynomals* J_λ are defined as the unique family satisfying the following relations:

- Orthogonality: $\langle J_\lambda, J_\mu \rangle_\alpha = 0$ whenever $\lambda \neq \mu$.
- Triangularity: $J_\lambda = \sum_{\mu \trianglelefteq \lambda} c_{\lambda\mu} m_\mu$
- Normalization: $[m_{1^n}]J_\lambda = n!$.

Let $a_\lambda(i,j)$ and $l_\lambda(i,j)$ be the *arm length* and *leg length* of a cell $(i,j) \in \lambda$, i.e., the number of cells in row i to the right of (i,j), and the number of cells in column j below (i,j).

Let $h^*_\lambda(i,j) := a_\lambda(i,j)\alpha + l_\lambda(i,j) + 1$ be the *lower hook length*. Let H^*_T be the product of the lower hook lengths of a shape T.

Specht Polynomials and Differential Operators

Let $\{f_{s,t} \in \mathbb{R}[X] : s,t \text{ standard } \lambda\text{-tableaux}, \lambda \vdash n\}$ be the *Specht polynomial basis* of $\mathbb{R}S_n$ defined such that

$$f_{s,t}(X) := \sum_{\tau \in R_s} \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) X(\tau s, \sigma t), \qquad X(s,t) := \prod_{(i,j) \in \lambda} X_{s_{i,j}, t_{i,j}}$$

where C_t (R_t) is the *column (row) stabilizer* of t. They are a sum of λ_1-many products of determinants corresponding to the pairs of columns of s,t.

What is the canonical presentation of each $f_{s,t} \in \mathbb{R}S_n$?

Let $I = i_1, \ldots, i_d \in [n]$ be distinct and $J = j_1, \ldots, j_d \in [n]$ be distinct. Let X be the $d \times d$ matrix with $X_{a,b} = X_{i_a, j_b}$. Define the *quasi-determinant* as

$$q(I,J)(X) := \sum_{i \in I, j \in J} \frac{\partial}{\partial X_{i,j}} \det X = \sum_{i \in I, j \in J} \frac{\partial}{\partial X_{i,j}} \sum_{\pi \in S_d} \operatorname{sgn}(\pi) \prod_{s \in [d]} X_{i_s, j_{\pi(s)}}.$$

Let $f'_{s,t}(X)$ be the *quasi-Specht polynomials* defined such that

$$f'_{s,t}(X) := \sum_{\tau \in R_s} \prod_{i=1}^{\lambda_1} q((\tau s)_i, t_i).$$

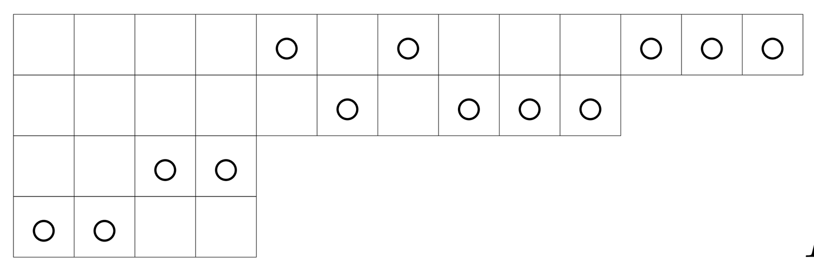
The $f'_{s,t}(X)$'s are harmonic. Define the differential operator

$$D_k := \Big(\sum_{i,j=1}^n \partial/\partial X_{i,j}\Big)^k / k!.$$

Theorem 2 *Let s,t be standard Young tableaux of shape λ. The canonical presentation of $f_{s,t}$ is $p_{s,t}(X) := d_\lambda(1)^{-1} f'_{s,t} = d_\lambda(1)^{-1} D_{\lambda_1} f_{s,t}$ where $d_\lambda(1)$ is the product of the hook lengths along the top row of λ.*

We show the product of the lower hook lengths along the top row of λ are equal to weighted sums of so-called tableau traversals of λ.

A *tableau transversal* T of λ is a set of cells that forms a transversal of the columns of λ, e.g.,



$$H^*_T = \prod_{i=0}^{4} (i\alpha + 1) \prod_{i=0}^{3} (i\alpha + 1)\, (\alpha + 1)^2$$

Let $w_\alpha(\lambda) := \sum_T H^*_T$ where T ranges over all tableau transversals of λ.

Theorem 3 *For all λ, we have $\prod_{j=1}^{\lambda_1} h^*_\lambda(1,j) = w_\alpha(\lambda)$.*