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Lattice Paths And Negatively Indexed Weight-Dependent Binomial Coefficients

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Weight-dependent binomial coefficients

For $n, k \in \mathbb{Z}$, we define the weight-dependent binomial coefficient as

$$ {}_w\begin{bmatrix} n \\ 0 \end{bmatrix} = {}_w\begin{bmatrix} n \\ n \end{bmatrix} = 1 \qquad \text{for } n \in \mathbb{Z}\,, $$

and for $n, k \in \mathbb{Z}$, provided that $(n+1,k) \neq (0,0)$,

$$ {}_w\begin{bmatrix} n+1 \\ k \end{bmatrix} = {}_w\begin{bmatrix} n \\ k \end{bmatrix} + {}_w\begin{bmatrix} n \\ k-1 \end{bmatrix} W(k, n+1-k), $$

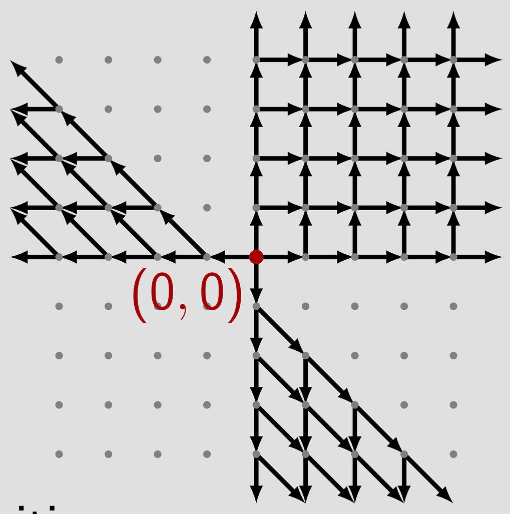
with $W(s,t) = \prod_{j=1}^{t} w(s,j)$ for a sequence of weights $w(s,t)$ and we define products generally by

$$ \prod_{j=1}^{t} A_j = \begin{cases} A_1 A_2 \dots A_t & t > 0 \\ 1 & t = 0 \\ A_0^{-1} A_{-1}^{-1} \dots A_{t+1}^{-1} & t < 0 \end{cases}. $$

The lattice path model

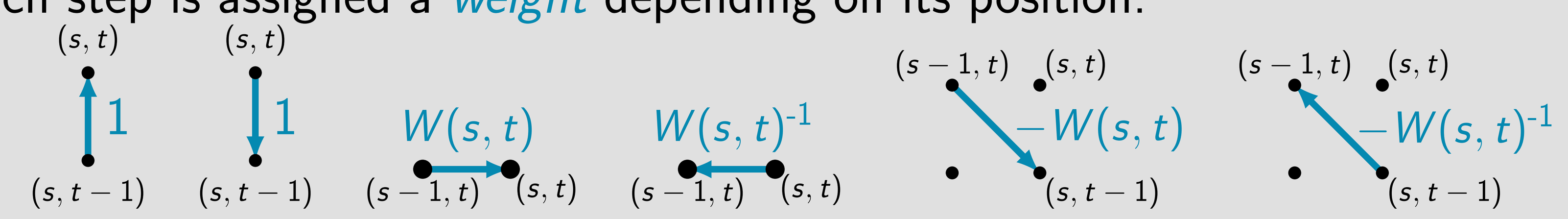
For $m, n \in \mathbb{Z}$, a *hybrid lattice path* is a path from $(0,0)$ to (m,n). Depending on m and n, the possible steps of a path are:

1. $n, m \geq 0$: $\uparrow$ and $\rightarrow$
2. $m < 0 \leq n$: $\leftarrow$ and $\nwarrow$
3. $n < 0 \leq m$: $\downarrow$ and $\searrow$
4. $n, m < 0$: no allowed steps



Additionally, if $m < 0$, the first step has to be $\leftarrow$ and if $n < 0$, the first step has to be $\downarrow$.

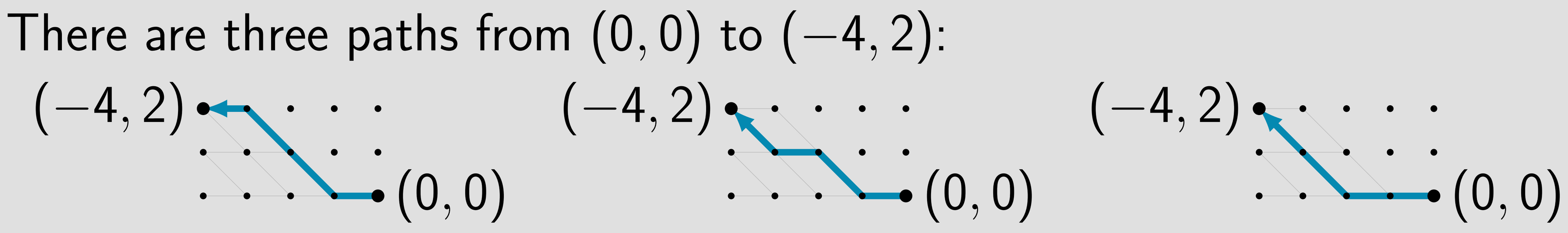
Each step is assigned a *weight* depending on its position:



The *weight of a path* $w(P)$ is the product of the weights of its steps.

Example

There are three paths from $(0,0)$ to $(-4,2)$:



The weight of the first path is for example

$$ \begin{aligned} w(P) &= W(0,0)^{-1}(-W(-1,1)^{-1})(-W(-2,2)^{-1})W(-3,2)^{-1} \\ &= (w(-1,1)w(-2,1)w(-2,2)w(-3,1)w(-3,2))^{-1}. \end{aligned} $$

Weighted counting

The weight-dependent binomial coefficients count weighted hybrid lattice paths. For all $n, k \in \mathbb{Z}$,

$$ {}_w\begin{bmatrix} n \\ k \end{bmatrix} = \sum_P w(P), $$

where the sum runs over all paths from $(0,0)$ to $(k, n-k)$.

Reflection formulae

We define the weight-reflections

$\widehat{w}(s,t) = w(t,s)^{-1}$, $\check{w}(s,t) = w(s,1-s-t)^{-1}$, $\widetilde{w}(s,t) = w(1-s-t,t)^{-1}$,

and $\operatorname{sgn}(n)$ to be 1 for $n \geq 0$ and -1 for $n < 0$, to obtain

$$ \begin{aligned} {}_w\begin{bmatrix} n \\ k \end{bmatrix} &= {}_{\widehat{w}}\begin{bmatrix} n \\ n-k \end{bmatrix} \prod_{j=1}^{k} W(j, n-k) \\ &= {}_{\check{w}}\begin{bmatrix} k-n-1 \\ k \end{bmatrix} (-1)^k \operatorname{sgn}(k) \prod_{j=1}^{k} W(j,-j) \\ &= {}_{\widetilde{w}}\begin{bmatrix} -k-1 \\ -n-1 \end{bmatrix} (-1)^{n-k} \operatorname{sgn}(n-k) \prod_{j=1}^{n-k} W(n+1-j, j)^{-1}. \end{aligned} $$

FWF Der Wissenschaftsfonds.

NRF National Research Foundation of Korea

Noncommutative binomial theorem

Let x and y be noncommutative variables satisfying the three relations

$$ yx = w(1,1)xy, \; xw(s,t) = w(s+1,t)x \text{ and } yw(s,t) = w(s,t+1)y, $$

for all $s, t \in \mathbb{Z}$, then for all $n \in \mathbb{Z}$:

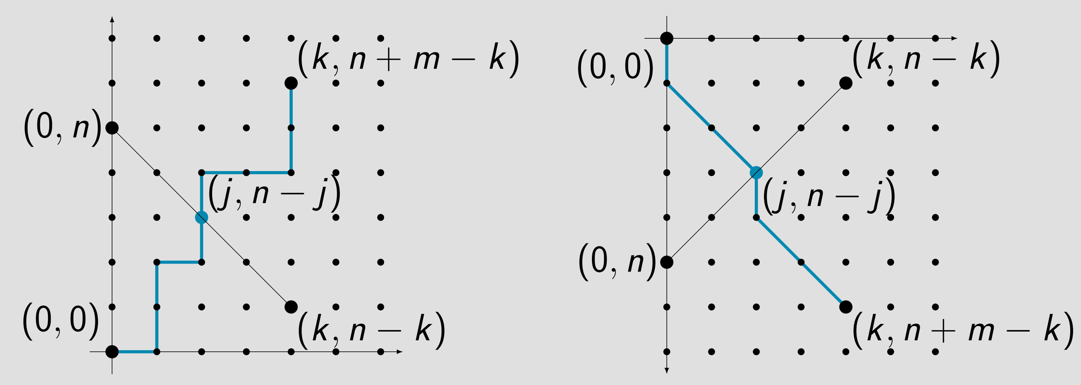
$$ (x+y)^n = \sum_{k \geq 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k} \quad \text{or} \quad (x+y)^n = \sum_{k \leq n} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. $$

Convolution formula

Let x, y be noncommutative as before, $n, m \in \mathbb{Z}$ and $k \geq 0$, then

$$ {}_w\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{j=0}^{k} {}_w\begin{bmatrix} n \\ j \end{bmatrix} \left(x^j y^{n-j} {}_w\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). $$

For $m, n > 0$ or $m, n < 0$, this identity can be interpreted as convolution over weighted paths with respect to a diagonal.



Specializations

Binomial coefficient

For $w(s,t) = 1$ we obtain the ordinary *binomial coefficient* $\binom{n}{k}$ studied for arbitrary integer values by Loeb [1].

Gaussian binomial coefficient

For $w(s,t) = q$ we obtain the *q-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ studied for arbitrary integer values by Formichella and Straub [2].

Elliptic binomial coefficient

For

$$ w(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b; p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b; p)} q $$

we obtain the *elliptic binomial coefficient* [3]

$$ {}_w\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}}, $$

where $\theta(x;p) = \prod_{k=0}^{\infty}((1-xp^k)(1-p^{k+1}/x))$ is the *modified Jacobi theta function* and $(a;q,p)_k = \prod_{i=0}^{k-1} \theta(aq^i;p)$ is the *theta shifted factorial*.

Symmetric functions

For $w(s,t) = \frac{a_{s+t}}{a_{s+t-1}}$ we obtain

$$ {}_w\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} e_k(a_1, a_2, \dots, a_n) \prod_{i=1}^{k} a_i^{-1}, & 0 \leq k \leq n \\ h_k(-a_0, -a_{-1}, \dots, -a_{n+1}) \prod_{i=1}^{k} a_i^{-1}, & n < 0 \leq k \\ h_{n-k}(-a_0^{-1}, -a_{-1}^{-1}, \dots, -a_{n+1}^{-1}) \prod_{i=k+1}^{n} a_i, & k \leq n < 0 \end{cases} $$

where e_k is the *elementary symmetric function* and h_k is the *complete homogeneous symmetric function* of order k.

Conclusion

- Many results from [1] and [2] can be generalised to the weighted case.
- In [1] and [2] binomial coefficients were interpreted with hybrid sets. Hybrid lattice paths can be translated to the corresponding hybrid sets.
- For more details see [4].

References

[1] Loeb, D. E.: *Sets with a negative number of elements.* Adv. Math., 91(1):64–74, 1992.

[2] Formichella, S.; Straub, A.: *Gaussian binomial coefficients with negative arguments.* Ann. Comb., 23(3-4):725–748, 2019.

[3] Schlosser, M. J.: *A noncommutative weight-dependent generalisation of the binomial theorem.* Sém. Lothar. Combin., 81:Art. B81j, 24, 2020.

[4] Küstner, J.; Schlosser, M. J.; Yoo, M.: *Lattice paths and negatively indexed weight-dependent binomial coefficients.* arXiv:2204.05505, 2022.