The elliptic Hall algebra element $\mathbf{Q}^k_{m,n}(1)$

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Torus link homology

Links and link invariants

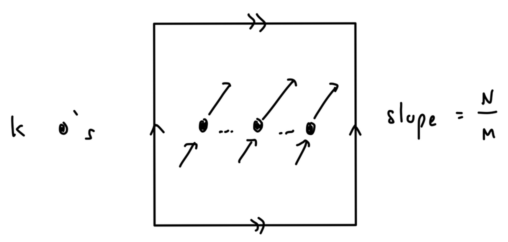
A *link* is a topological subspace of $\mathbb{R}^3$ whose connected components are homeomorphic to circles (they are *knots*). A *link invariant* is a map from the space of links that is invariant under ambient isotopy ("continuous distortion.") **Open question**: Is there a (nontrivial) "perfect" link invariant?

Khovanov–Rozansky homology

This is a powerful link invariant that assigns a triply-graded vector space $\bigoplus_{i,j,k\in\mathbb{Z}} V_{i,j,k}$ to each link. As this construction is complicated, it is often more manageable to study the related generating function $\sum_{i,j,k\in\mathbb{Z}} q^i t^j a^k \dim V_{i,j,k}$.

Torus links

Given $M, N \in \mathbb{Z}_{>0}$, with $k := \gcd(M,N)$, the M,N-*torus link* $T(M,N)$ consists of k knots wrapped around the torus in the following way:



Hogancamp and Mellit's recursion

Hogancamp and Mellit prove that computing $p(0^M, 0^N)$ via the following recursion produces the generating function for the KR homology of $T(M,N)$:

(i) $p(\bullet^M, \bullet^N) = 1$.

(ii) $p(\bullet v, \bullet w) = p(v\bullet, w\bullet)$.

(iii) $p(0v, 0w) = (1-q)^{-1} p(v1, w1)$ if $|v| = \#1\text{'s in } v = |w| = 0$.

(iv) $p(0v, 0w) = t^{-\ell} p(v1, w1) + qt^{-\ell} p(v0, w0)$ if $\ell = |v| = |w| > 0$.

(v) $p(1v, 0w) = p(v1, w\bullet)$.

(vi) $p(0v, 1w) = p(v\bullet, w1)$.

(vii) $p(1v, 1w) = (t^{|v|} + a) p(v\bullet, w\bullet)$.

An example

For $T(1,2)$, we get

$$\begin{aligned}
p(0,00) &= (1-q)^{-1} p(1,01) && \text{(iii)}\\
&= (1-q)^{-1} p(1,1\bullet) && \text{(v)}\\
&= (1-q)^{-1} (1+a) p(\bullet,\bullet\bullet) && \text{(vii)}\\
&= (1-q)^{-1} (1+a) && \text{(i)}
\end{aligned}$$

In fact, if M or N is 1, $T(M,N)$ is trivial and we should always get this answer.

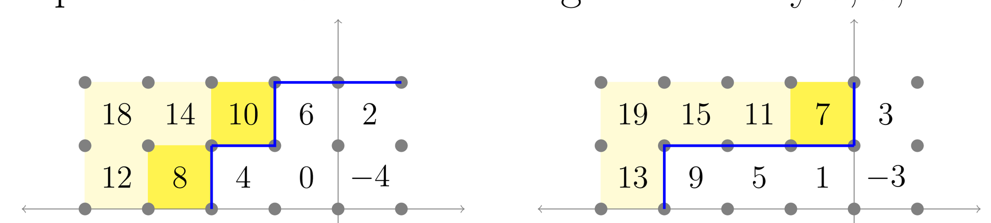
Lifting to symmetric functions

Invariant sets

An *invariant set* is a subset $I \subseteq \mathbb{Z}_{\geq 0}$ with finite complement such that

$$i \in I \implies i+M,\ i+N \in I.$$

There is a bijection between invariant sets and k-tuples of Dyck path-like objects. Here is an example for $M = 6$ and $N = 4$ "generated" by 7, 8, and 10:



"Defining" the symmetric function lift

We define a symmetric function $L_{M,N}$ as a sum over invariant sets I:

$$L_{M,N} := \sum_I q^{\mathrm{area}(I)} t^{-\mathrm{dinv}(I)} \chi(I)$$

for certain statistics area and dinv and where $\chi(I)$ is an "LLT polynomial."

A recursion for $L_{M,N}$

Our main result is that $L_{M,N} = L(0^M, 0^N)$ as computed by the following recursion. This specializes to Hogancamp and Mellit's recursion at $e_i \mapsto a+1$ for all i.

(I) $L(\bullet^M, \bullet^N) = 1$.

(II) $L(\bullet v, \bullet w) = L(v\bullet, w\bullet)$.

(III) $L(0v, 0w) = (1-q)^{-1} d_- L(v1, w1)$ if $|v| = |w| = 0$.

(IV) $L(0v, 0w) = t^{-\ell} d_- L(v1, w1) + qt^{-\ell} L(v0, w0)$ if $\ell = |v| = |w| > 0$.

(V) $L(1v, 0w) = t^{-\ell} d_= L(v1, w\bullet)$ if $\ell = |v| = |w| - 1$.

(VI) $L(0v, 1w) = L(v\bullet, w1)$.

(VII) $L(1v, 1w) = d_+ L(v\bullet, w\bullet)$

d_+, d_-, and $d_=$ are creation operators for LLT polynomials, introduced by Carlsson and Mellit in their proof of the Shuffle Theorem.

Continuing the example

Again assume $M = 1$ and $N = 2$. Then

$$\begin{aligned}
L(0,00) &= (1-q)^{-1} d_- L(1,01) && \text{(III)}\\
&= (1-q)^{-1} d_- d_= L(1,1\bullet) && \text{(V)}\\
&= (1-q)^{-1} d_- d_= d_+ L(\bullet,\bullet\bullet) && \text{(VII)}\\
&= (1-q)^{-1} d_- d_= d_+(1) && \text{(I)}.
\end{aligned}$$

The LLT polynomial corresponding to $d_- d_= d_+(1)$ is $e_2 = s_{1,1}$.

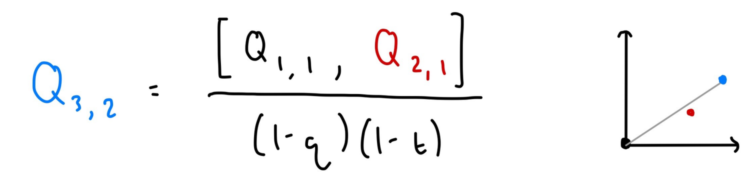
A conjecture for $L_{M,N}$

The $Q_{m,n}$ operator

Given $m, n \in \mathbb{Z}_{>0}$ with $\gcd(m,n) = 1$, one can define a recursive operator $\boldsymbol{Q}_{m,n}$ on a symmetric function f by

$$\boldsymbol{Q}_{m,n}(f) := \frac{1}{(1-q)(1-t)} [\boldsymbol{Q}_{m-a,n-b}, \boldsymbol{Q}_{a,b}](f)$$

where (a,b) is the closest lattice point below the line from $(0,0)$ to (m,n). (We also have $\boldsymbol{Q}_{1,n} := \boldsymbol{D}_n$, a certain plethystic Macdonald operator.)



The conjecture

For positive integers M and N, let $k = \gcd(M,N)$, $m = M/k$, $n = N/k$. Then

$$\mathbf{Q}^k_{m,n}(1) = \pm(1-q)^k t^C L_{M,N}$$

where C is the maximum of dinv over all invariant sets.

Special cases of the conjecture

- $\boldsymbol{Q}_{n+1,n}(1) \approx \nabla e_n$ and we recover the Shuffle Theorem.
- More generally, if $k = 1$ we recover the Rational Shuffle Theorem.
- If $N = n$ and $M = kn$, we recover an open conjecture for $\nabla^k e_{1^n}$.

Open problems

- Prove the conjecture! Maybe using methods from Blasiak et. al.?
- Extend to non-torus links, e.g. Galashin and Lam's "positroid links."