Grothendieck-to-Lascoux expansions

Mark Shimozono [1] Tianyi Yu [2]

[1]Virginia Tech [2]University of California, San Diego

Four operators on polynomials

The group $S_+ = \bigcup_{n\geq 1} S_n$ acts on $R = \mathbb{Z}[\beta][x_1, x_2, \dots]$ by permuting the variables: s_i exchange x_i and x_{i+1}. Define the following operators on R, where an element $f \in R$ (or its fraction field) denotes the operator of left multiplication by f.

$$\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i)$$
$$\pi_i = \partial_i x_i$$
$$\partial_i^{(\beta)} = \partial_i(1 + \beta x_{i+1})$$
$$\pi_i^{(\beta)} = \partial_i^{(\beta)} x_i.$$

All satisfy the braid relations for S_+.

Eight polynomials

Let $w_0^{(n)} \in S_n$ be the long element. For $w \in S_n$, the **Grothendieck polynomial** is defined by [LasSc]

$$\mathfrak{G}_w^{(\beta)} = \begin{cases} x_1^{n-1}x_2^{n-2}\dots x_{n-1} & \text{if } w = w_0^{(n)} \\ \partial_i^{(\beta)}\mathfrak{G}_{ws_i}^{(\beta)} & \text{if } ws_i > w. \end{cases}$$

The **Schubert polynomial** $\mathfrak{S}_w$ is defined by

$$\mathfrak{S}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=0}.$$

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a composition (sequence of nonnegative integers, almost all 0). The **Lascoux polynomial** $\mathfrak{L}_\alpha^{(\beta)}$ is defined by [Las]

$$\mathfrak{L}_\alpha^{(\beta)} = \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition} \\ \pi_i^{(\beta)}\mathfrak{L}_{s_i\alpha}^{(\beta)} & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

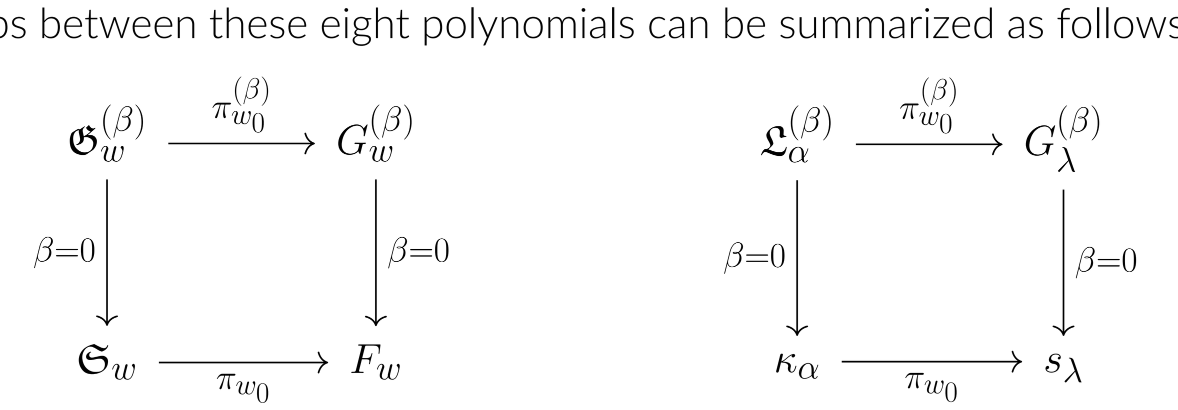
The **Demazure character** κ_α is defined by

$$\kappa_\alpha = \mathfrak{L}_\alpha^{(\beta)}|_{\beta=0}.$$

Given a composition $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ let α^+ be the unique partition in the S_n-orbit of α. We may symmetrize the four polynomials above via operators $\pi_{w_0}^{(\beta)}$ and π_{w_0}. We get four polynomials that are symmetric in the x variables.

$\pi_{w_0}^{(\beta)}(\mathfrak{L}_\alpha) = G_{\alpha^+}^{(\beta)}(x_1, \dots, x_n)$	**Grassmannian Grothendieck symmetric functions**
$\pi_{w_0}(\kappa_\alpha) = s_{\alpha^+}(x_1, \dots, x_n)$	**Schur functions**
$\pi_{w_0}^{(\beta)}(\mathfrak{G}_w(x)) = G_w(x_1, \dots, x_n)$	**Grothendieck symmetric functions**
$\pi_{w_0}(\mathfrak{S}_w(x)) = F_w(x_1, \dots, x_n)$	**Stanley symmetric functions**

The relationships between these eight polynomials can be summarized as follows.



Each polynomial in the first diagram can be expanded into corresponding polynomials in the second diagram. The focus of this poster is expanding $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$.

Hecke words

Let be $\mathbb{Z}_{>0}^*$ be the free monoid of words in alphabet $\mathbb{Z}_{>0}$.

$\mathbb{Z}_{>0}^*$ acts on S_+:

$$i \circ w = \begin{cases} s_i w & \text{if } \ell(s_i w) > \ell(w). \\ w & \text{otherwise.} \end{cases}$$

Let $[b]_H := b \circ \mathrm{id} \in S_+$. For instance, $[3124]_H = [3142]_H = [31424]_H = 31524$.

Increasing and Decreasing tableaux

A tableau is increasing (resp. decreasing) if each row and column is strictly increasing (resp. decreasing).

The column reading word of a tableau P, denoted as $\mathrm{col}(P)$, is obtained by reading the entries of P from left to right and bottom to top within each column.

Each increasing tableau P is associated with a weak composition, denoted as $K_-(P)$. Each decreasing tableau P is associated with a weak composition, denoted as $K_+(P)$. They can be computed using K-theoretic jeu-de-taquin of Thomas and Yong [TY].

Grothendieck-to-Lascoux expansions (decreasing version)

Our main result is the following. Let w be a permutation. Then its Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}$ can be written as

$$\mathfrak{G}_w^{(\beta)} = \sum_P \mathfrak{L}_{K_+(P)}^{(\beta)} \beta^{\mathrm{sum}(K_+(P)) - \ell(w)},$$

where the sum is over all **decreasing tableaux** P such that $[\mathrm{rev}(\mathrm{col}(P))]_H = w$.

For instance, when $w = 31524$, P can be:

4	3	1
2		

4	3
2	1

4	3	1
2	1	

Then $K_+(P)$ are 301, 202, and 302. Thus, $\mathfrak{G}_{31524}^{(\beta)} = \mathfrak{L}_{301}^{(\beta)} + \mathfrak{L}_{202}^{(\beta)} + \beta\mathfrak{L}_{302}^{(\beta)}$.

Grothendieck-to-Lascoux expansions (increasing version)

Reiner and Yong conjecture [ReY]:

$$\mathfrak{G}_w^{(\beta)} = \sum_P \mathfrak{L}_{K_-(P)}^{(\beta)} \beta^{\mathrm{sum}(K_+(P)) - \ell(w)},$$

where the sum is over all **increasing tableaux** P such that $[\mathrm{col}(P)]_H = w$.

For instance, when $w = 31524$, P can be:

1	2	4
3		

1	2
3	4

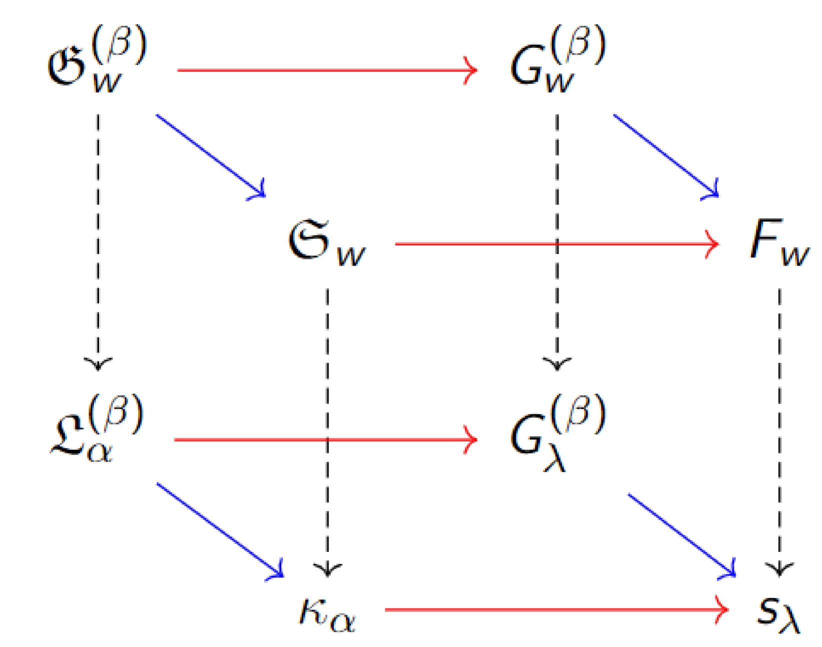
1	2	4
3	4	

Then $K_-(P)$ are 301, 202, and 302. Thus, $\mathfrak{G}_{31524}^{(\beta)} = \mathfrak{L}_{301}^{(\beta)} + \mathfrak{L}_{202}^{(\beta)} + \beta\mathfrak{L}_{302}^{(\beta)}$, which agrees with the previous version.

We established this conjecture by building a bijection from decreasing tableaux to increasing tableaux: $P \mapsto P^\sharp$. It satisfies $K_+(P) = K_-(P^\sharp)$ and $[\mathrm{rev}(\mathrm{col}(P))]_H = [\mathrm{col}(P^\sharp)]_H$.

Four expansions

Our expansion fits into a family of four expansions, involving the eight polynomials. In the following picture, the four dashed arrows represent expansions. Red arrows represent symmetrization and blue arrows represent setting $\beta = 0$.



- F_w into s_λ: Edelman and Greene [EG] established this expansion via a Schensted-type insertion algorithm for reduced words, called Edelman-Greene insertion.
- $\mathfrak{S}_w$ into κ_α: The expansion was found by Lascoux and Schützenberger and proved in [RS].
- $G_w^{(\beta)}$ into $G_\lambda^{(\beta)}$: This expansion was established by Buch, Kresch, Shimozono, Tamvakis, and Yong [BKSTY] via Hecke insertion. Hecke insertion takes Hecke words as input, generalizing the Edelman-Greene insertion.
- $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$: This expansion is the topic of this poster. It is also established using the Hecke insertion.

References

[BKSTY] A. Buch, A. Kresch, M. Shimozono, H. Tamvakis, and A. Yong. Stable Grothendieck polynomials and K-theoretic factor sequences. Math. Ann. 340 (2008), no. 2, 359–382.

[EG] P. Edelman and C. Greene. Balanced tableaux. Adv. in Math. 63 (1987), no. 1, 42–99.

[Las] A. Lascoux. Schubert & Grothendieck: un bilan bidécennal, Sém. Lothar. Combin. 50 (2003/04), Art. B50i.

[LasSc] A. Lascoux and M. P. Schützenberger. Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux. C. R. Acad. Sci. Paris Sér. I Math. 295, 11 (1982), 629–633.

[ReY] V. Reiner and A. Yong. The "Grothendieck-to-Lascoux" conjecture. preprint arXiv:2102.12399v1, 2021.

[RS] V. Reiner and M. Shimozono. Key polynomials and a flagged Littlewood-Richardson rule. J. Combin. Theory Ser. A 70 (1995), no. 1, 107–143.

[TY] H. Thomas and A. Yong. A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus. Algebra Number Theory 3 (2009), no. 2, 121–148.