Refined Consecutive Pattern Enumeration via a Generalized Cluster Method

Yan Zhuang

Department of Mathematics and Computer Science, Davidson College

DAVIDSON

Abstract

We present a general approach for counting permutations by occurrences of prescribed consecutive patterns together with various inverse statistics. We first lift the Goulden–Jackson cluster method for permutations—a standard tool in the study of consecutive patterns—to the Malvenuto–Reutenauer algebra. Upon applying homomorphisms, our result specializes to both the cluster method for permutations as well as a q-analogue which keeps track of the inversion number statistic. We construct additional homomorphisms which lead to further specializations for keeping track of inverses of shuffle-compatible permutation statistics; these include the inverse descent number, inverse peak number, and inverse left peak number. To illustrate this approach, we present new formulas that count permutations by occurrences of the monotone consecutive pattern $12\cdots m$ while also keeping track of these inverse statistics.

Background: Consecutive Patterns in Permutations

- Let $\mathfrak{S} = \bigcup_{n=0}^{\infty} \mathfrak{S}_n$. We say that $\pi \in \mathfrak{S}$ *contains* $\sigma \in \mathfrak{S}$ (as a *consecutive pattern*) if π has a consecutive subsequence w with the same relative order as σ. Then w is called an *occurrence* of σ.
- **Example:** Let $\pi = 6351427$. Then $w = 351$ is an occurrence of $\sigma = 231$ in π.
- Let $\mathrm{occ}_\sigma(\pi)$ be the number of occurrences of σ in π.
- **Big Question:** How can we count permutations by the number of occurrences of a prescribed consecutive pattern?

Background: The Goulden–Jackson Cluster Method

- Given $\sigma \in \mathfrak{S}$, a σ-*cluster* is a permutation filled with marked occurrences of σ that overlap with each other.
- **Example:** An example of a 1324-cluster: 1 4 2 5 3 6 8 7 9

  Two non-examples: 1 4 3 6 2 7 5 8, 4 2 6 3 8 5 9 1 7
- Let $C_{\sigma,\pi}$ be the set of σ-clusters with underlying permutation π.
- Let $\mathrm{mk}_\sigma(c)$ be the number of marked occurrences of σ in the σ-cluster c.
- **Example:** If c is the 1324-cluster in the above example, then $c \in C_{1324,142536879}$ and $\mathrm{mk}_{1324}(c) = 3$.
- **Theorem** (Cluster method for permutations)**:** For $\sigma \in \mathfrak{S}$, let
$$F_\sigma(s,x) = \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{occ}_\sigma(\pi)} \frac{x^n}{n!} \quad \text{and} \quad R_\sigma(s,x) = \sum_{n=2}^{\infty} \sum_{\pi \in \mathfrak{S}_n} \sum_{c \in C_{\sigma,\pi}} s^{\mathrm{mk}_\sigma(c)} \frac{x^n}{n!}.$$
  Then, if σ has length at least 2, we have
$$F_\sigma(s,x) = \left(1 - x - R_\sigma(s-1,x)\right)^{-1}. \qquad (\heartsuit)$$
  (The original cluster method of Goulden and Jackson [4] was for words; Elizalde and Noy [2] adapted it for permutations.)
- **What this means:** We can count permutations by occurrences of σ if we can count σ-clusters by marked occurrences!

Lifting the Cluster Method to Malvenuto–Reutenauer

- Let $\mathbb{Q}[\mathfrak{S}]$ be the $\mathbb{Q}$-vector space with basis $\mathfrak{S}$. The *Malvenuto–Reutenauer algebra* is the $\mathbb{Q}$-algebra on $\mathbb{Q}[\mathfrak{S}]$ with multiplication
$$\pi \cdot \sigma = \sum_{\tau \in C(\pi,\sigma)} \tau$$
  where $C(\pi,\sigma)$ is the set of all *concatenations* of π and σ.
- **Example:** In $\mathbb{Q}[\mathfrak{S}]$, $12 \cdot 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421$.
- **Main Result** (Cluster method in Malvenuto–Reutenauer)**:** For $\sigma \in \mathfrak{S}$, let
$$F_\sigma(s) = \sum_{\pi \in \mathfrak{S}} \pi s^{\mathrm{occ}_\sigma(\pi)} \quad \text{and} \quad R_\sigma(s) = \sum_{\pi \in \mathfrak{S}} \sum_{c \in C_{\sigma,\pi}} \pi s^{\mathrm{mk}_\sigma(c)}.$$
  Then, if σ has length at least 2, we have
$$F_\sigma(s) = \left(\varepsilon - \iota - R_\sigma(s-1)\right)^{-1}, \qquad (\spadesuit)$$
  where ε is the empty permutation and ι the permutation of length 1.

- Let $\Phi(\pi) = x^n/n!$ where n is the length of π. Applying Φ to ($\spadesuit$) recovers the cluster method for permutations ($\heartsuit$).
- Let $\Phi_q(\pi) = q^{\mathrm{inv}(\pi)} x^n/[n]_q!$ where n is the length of π and $\mathrm{inv}(\pi)$ is the number of *inversions* of π. Applying Φ_q to ($\spadesuit$) instead recovers a q-analogue of ($\heartsuit$) that refines by inv [1].
- **Big Question:** Are there other homomorphisms on $\mathbb{Q}[\mathfrak{S}]$ that lead to specializations of ($\spadesuit$) which refine by other statistics?

A Foray into Shuffle-Compatibility

- Let π and σ be permutations on disjoint sets of positive integers, and let $S(\pi,\sigma)$ be the set of *shuffles* of π and σ.
- **Example:** $S(13,42) = \{1342, 1432, 1423, 4213, 4123, 4132\}$.
- A permutation statistic st is *shuffle-compatible* if the distribution of st over $S(\pi,\sigma)$ depends only on $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, and the lengths of σ and π.
- Let $\mathrm{Des}(\pi)$ be the descent set of π. Then st is a *descent statistic* if $\mathrm{Des}(\pi) = \mathrm{Des}(\sigma)$ and $|\pi| = |\sigma|$ imply $\mathrm{st}(\pi) = \mathrm{st}(\sigma)$.
- If st is a permutation statistic, let $\mathrm{ist}(\pi) = \mathrm{st}(\pi^{-1})$ be its *inverse statistic*.
- **General Principle:** If st is a shuffle-compatible descent statistic, then there is a homomorphism on $\mathbb{Q}[\mathfrak{S}]$ for counting permutations by ist.
- Examples of shuffle-compatible descent statistics: the *descent number* des, *major index* maj, *peak number* pk, and *left peak number* lpk [3]. So we have homomorphisms for the corresponding inverse statistics!

New Specializations and Explicit Formulas

1. Specializations of the generalized cluster method ($\spadesuit$) for ides, ipk, and ilpk. **This gives a general approach for counting permutations by each of these statistics jointly with occurrences of a consecutive pattern!**
2. Generating function formulas for counting permutations by ides, ipk, and ilpk along with occurrences of these patterns:
   - $12\cdots m$ and $m\cdots 21$ for $m \geq 2$;
   - $12\cdots(a-1)(a+1)a(a+2)(a+3)\cdots m$ for $m \geq 5$ and $2 \leq a \leq m-2$;
   - $2134\cdots m$ and $12\cdots(m-2)m(m-1)$ for $m \geq 3$ (in progress; ongoing work with Sergi Elizalde and Justin Troyka).

(Extended abstract contains results for $12\cdots m$; see [5] for full results.)

A Sample Formula

- The *Hadamard product* $*$ on formal power series in t is defined by
$$\left(\sum_{n=0}^{\infty} a_n t^n\right) * \left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty} a_n b_n t^n.$$
- Let $f^{*\langle n\rangle} = \underbrace{f * f * \cdots * f}_{n \text{ copies of } f}$ be the n-fold Hadamard product of $f \in \mathbb{Q}[[t]]$.
- For $\sigma \in \mathfrak{S}$, let $A^{\mathrm{ides}}_{\sigma,n}(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{occ}_\sigma(\pi)} t^{\mathrm{ides}(\pi)+1}$ for $n \geq 1$ and $A^{\mathrm{ides}}_{\sigma,0}(s,t) = 1$.
- **Theorem:** For all $m \geq 2$, we have
$$\sum_{n=0}^{\infty} \frac{A^{\mathrm{ides}}_{12\cdots m,n}(s,t)}{(1-t)^{n+1}} x^n = \sum_{n=0}^{\infty} \left(\frac{tx}{(1-t)^2} + \frac{(s-1)ty^m(y-1)}{(t-1)(y^m(s-1)-ys+1)} \right)^{*\langle n\rangle}$$
  where $y = x/(1-t)$. (Here, the Hadamard product is only in the variable t. Multiplication in the other variables is ordinary multiplication.)

References

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