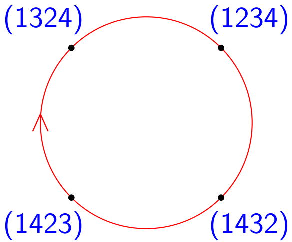
Cyclic Descent Extensions and Higher Lie Characters

Ron Adin (Bar-Ilan), Pál Hegedüs (Renyi Inst.), Yuval Roichman (Bar-Ilan)



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Further studied by Fulman ['00], Petersen ['05, '07], Dilks-Petersen-Stembridge ['09], Rhoades ['10], Visontai-Williams ['13], Zhang ['14], Pechenik ['14], Aguiar-Petersen ['15], Elizalde-R ['17], Ahlbach-Swanson ['18], Bloom-Elizalde-R ['20], Adin-Reiner-R ['20], Huang ['20], Bloom-Elizalde-R ['20], Zakeri ['21], Adin-Gessel-Reiner-R ['21], Khachatryan ['22] and others ...

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Define a cyclic descent set for SYT of any shape λ.

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$$\begin{aligned} \mathrm{cDes}(\pi) \cap [n-1] &= \mathrm{Des}(\pi), \\ \mathrm{cDes}(p(\pi)) &= \mathrm{cDes}(\pi) + 1 \pmod{n} \end{aligned}$$

where the rotation $p([\pi_1, \ldots, \pi_n]) := [\pi_n, \pi_1, \ldots, \pi_{n-1}]$.

SYT of rectangular shapes

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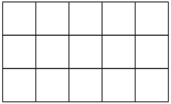

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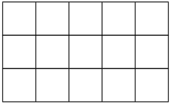

Theorem (Rhoades '10)

For $r|n$, *let* $\lambda = (r^{n/r}) = (r, \ldots, r) \vdash n$ *be a rectangular shape. Then there exists a cyclic descent map* $\mathrm{cDes} : \mathrm{SYT}(\lambda) \to 2^{[n]}$ *s.t. for all* $T \in \mathrm{SYT}(\lambda)$*:*

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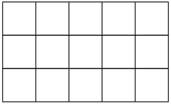

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where p *is Schützenberger's jeu-de-taquin promotion operator.*

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$$\{1,4\} \quad \{2,5\} \quad \{3,6\} \qquad\qquad \{1,3,5\} \quad \{2,4,6\}$$

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Examples

- $\mathcal{T} = S_n$, cDes = Cellini's cyclic descent set, and p = cyclic rotation.
- $\mathcal{T} = \mathrm{SYT}(r^{n/r})$, cDes = Rhoades' cyclic descent set, and p = promotion.

A non-Escher property

“Ascending and Descending”, M. C. Escher

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The paradox of $\mathrm{cDes}(\pi) = \varnothing$ and $\mathrm{cDes}(\pi) = [n]$.

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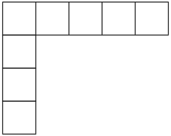

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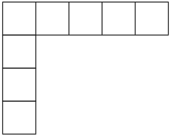

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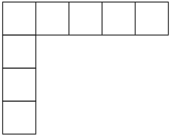

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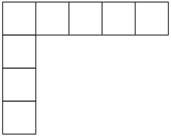

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- A constructive combinatorial proof was given by Brice Huang.

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- Geometric: Steinberg torus, Gromov-Witten invariants.

Cyclic descent extension on conjugacy classes

Let $\mathcal{C}_\mu \subseteq S_n$ be a conjugacy class of cycle type μ.

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The following are equivalent:

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Problem 3:

Find a constructive combinatorial proof.

Higher Lie characters

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$$\mathbf{L}_\mu := \sum_{\pi \in \mathcal{C}_\mu} \mathcal{F}_{n,\mathrm{Des}(\pi)},$$

where

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Gessel's fundamental quasi-symmetric function.

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Let Z_μ be the centralizer of $\pi \in \mathcal{C}_\mu$.
There exists a 1-dim character ω^μ of Z_μ such that

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The higher Lie character indexed by $\mu \vdash n$ is

$$\psi^\mu := \omega^\mu \uparrow_{Z_\mu}^{S_n} .$$

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KraskiewiczWeyman, DésarménienWachs, GesselReutenauer, Sundaram, Schocker, HershReiner, AhlbachSwanson...

Hook multiplicities and CDE

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A subset $\mathcal{A} \subseteq S_n$ is Schur-positive if the associated quasi-symmetric function

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Lemma (AHR)

A Schur-positive set $\mathcal{A} \subseteq S_n$ has a cyclic descent extension $\iff$ the following two conditions hold:

(divisibility) *the polynomial* $\sum_{k=0}^{n-1} \langle \mathcal{Q}(\mathcal{A}), s_{(n-k,1^k)} \rangle x^k$ *is divisible by* $1 + x$;

(non-negativity) *the quotient has nonnegative coefficients.*

Divisibility

Lemma

For every S_n-character ϕ, the hook-multiplicity generating function

$$M_\phi(x) := \sum_{k=0}^{n-1} \langle \phi, \chi^{n-k,1^k} \rangle x^k$$

is divisible by $1 + x$ if and only if the value of ϕ on an n-cycle is zero: $\phi_{(n)} = 0$.

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Lemma

For $\lambda \vdash n$

$$\psi^\lambda_{(n)} = \begin{cases} \mu(r), & \text{if } \lambda = (r^s); \\ 0, & \text{otherwise}, \end{cases}$$

where $\mu(r)$ is the Möbius function.

Non-negativity - the case of distinct cycle lengths

Lemma (AHR)

Let $\lambda = (r^s) \sqcup \nu$ be a partition of n, where ν is a partition of $n - rs$ with no part equal to r. Then

$$M_\lambda(x) := \sum_{k=0}^{n-1} \langle \mathbf{L}_\lambda, s_{(n-k,1^k)} \rangle x^k = (1+x) M_{(r^s)}(x) M_\nu(x).$$

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Corollary

For conjugacy classes $\mathcal{C}_\lambda$ with distinct cycle lengths, the hook multiplicities g.f. $M_\lambda(x)$ is divisible by $1+x$, and the quotient has non-negative coefficients.

Non-negativity - the n-cycle case

Denote

$$m_{k,\lambda} := \langle \mathbf{L}_\lambda, s_{(n-k,1^k)} \rangle.$$

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Lemma [Sundaram '94]

$$m_{k-1,(n)} + m_{k,(n)} = \langle \mathbf{L}_{(n)}, e_k h_{n-k} \rangle = \frac{1}{n} \sum_{d|(n,k)} \mu(d)(-1)^{k+k/d} \binom{n/d}{k/d}.$$

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Theorem (AHR)

For every positive integer n the sequence

$$m_{0,(n)}, m_{1,(n)}, \ldots, m_{n-1,(n)}$$

is unimodal.

Non-negativity - the case of cycle type $(r, \ldots, r)$

We have to prove that for every $s \geq 1$ and square-free r

$$\frac{M_{(r^s)}(x)}{1+x} := \frac{\sum_k \langle \mathbf{L}_{(r^s)}, s_{(n-k,1^k)} \rangle x^k}{1+x}$$

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Observation

$$\frac{E_r(x,y) - 1}{(1+x)^2} = \sum_{s \geq 1} y^s \frac{M_{(r^s)}(x)}{1+x}.$$

Lemma [Sundaram '94]

$$f_{r,k} := \langle \mathbf{L}_{(r)}, e_k h_{r-k} \rangle = \frac{1}{r} \sum_{d|(r,k)} \mu(d)(-1)^{k+k/d} \binom{r/d}{k/d}.$$

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Corollary For every $r \geq 1$

$$\sum_{k,s} \langle \mathbf{L}_{(r^s)}, e_k h_{r^s-k} \rangle x^k y^s = \prod_j (1 - (-1)^j x^j y)^{(-1)^{j+1} \langle \mathbf{L}_{(r)}, e_j h_{r-j} \rangle}.$$

Open Problems

Problem 1:

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Conjecture 2:

For every partition $\lambda \vdash n$, the g.f.

$$\sum_{k=0}^{n-1} \langle \mathbf{L}_\lambda, s_{(n-k,1^k)} \rangle x^k$$

is unimodal.

Thank You!