NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Awards Screening Test

JUNE 25, 2005

Time Allowed: 90 Minutes

Maximum Marks: 40

Please read, carefully, the instructions on the following page before you write anything on this booklet

NAME:	**ROLL No.:**
Institution	

(For Official Use)

Sec. 1	**Sec. 2**	**Sec. 3**	**Sec. 4**	**Sec. 5**	**TOTAL**

INSTRUCTIONS TO CANDIDATES

- Do not forget to write your name and roll number on the cover page. In the box marked 'Institution', fill in the name of the institution where you are working towards a Ph.D. degree. In case you have not yet joined any institution for research, write *Not Applicable.*

- Please ensure that your answer booklet contains 13 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.

- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum marks to be scored is **forty**.

- Answer each question, as directed, in the space provided at the end of it. Answers are to be given in the form of a word (or words, if required), a numerical value (or values) or a simple mathematical expression. **Do not write sentences.**

- In certain questions (Qns. 1.7 to 1.10, 2.7 to 2.10) you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), (c) and (d)).

- Points will be awarded in the above questions and in Questions 3.5 to 3.10 only if **all** the correct choices are made. There will be no partial credit.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order.

Section 1: Algebra

1.1 Find the value of $a \in \mathbb{Z}$ such that $2 + \sqrt{3}$ is a root of the polynomial

$$x^3 - 5x^2 + ax - 1.$$

Answer:

1.2 Let $V = \mathbb{R}^5$ and $W = \mathbb{R}^7$. Let $T : V \to W$ be a linear map. If $\mathcal{N}(T)$ denotes the null space of T and $\mathcal{R}(T)$ denotes its range, then

$$\dim \mathcal{N}(\mathrm{T}) + \dim \mathcal{R}(\mathrm{T}) \ = ?$$

Answer:

1.3 Let A be a 3×3 matrix whose eigenvalues are $-1, 1, 2$. Find α, β and γ such that

$$A^{-1} \ = \ \alpha A^2 + \beta A + \gamma I.$$

Answer: $\alpha =$ .... $\beta =$ .... $\gamma =$ ....

1.4 What is the number of groups of order 6 (upto isomorphism)?

Answer:

1.5 Let G be a cyclic group of order 10. For $a \in G$, let $< a >$ denote the subgroup generated by a. How many elements are there in the set

$$\{a \in G \mid \ < a >= G\}?$$

Answer:

1.6 Let $\alpha = 2^{\frac{1}{3}}$ and $\beta = 5^{\frac{1}{4}}$. Let L be the field obtained by adjoining α and β to $\mathbb{Q}$. What is the degree of the extension $[L : \mathbb{Q}]$?

Answer:

1.7 Pick out the matrices which are diagonalizable over $\mathbb{C}$:
(a) Any $n \times n$ unitary matrix with complex entries.
(b) Any $n \times n$ hermitian matrix with complex entries.
(c) Any $n \times n$ strictly upper triangular matrix with complex entries.
(d) Any $n \times n$ matrix with complex entries whose eigenvalues are real.

Answer:

1.8 Pick out the units in $\mathbb{Z}[\sqrt{3}]$.
(a) $-7 + 4\sqrt{3}$ (b) $5 + 3\sqrt{3}$ (c) $2 - \sqrt{3}$ (d) $-3 - 2\sqrt{3}$.

Answer:

1.9 Pick out the integral domains from the following list of rings:
(a) $\{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$.
(b) The ring of continuous functions from $[0, 1]$ into $\mathbb{R}$.
(c) The ring of complex analytic functions on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$.
(d) The polynomial ring $\mathbb{Z}[x]$.

Answer:

1.10 Pick out the abelian groups from the following list:
(a) Any group of order 4.
(b) Any group of order 36.
(c) Any group of order 47.
(d) Any group of order 49.

Answer:

Section 2: Analysis

2.1 Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) = ?$$

Answer:

2.2 What is the radius of convergence of the following series?

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Answer:

2.3 Let $k \in [0, \infty[$ be a real number. Define

$$f_k(t) = \begin{cases} t^k \sin \frac{1}{t}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

Let $A = \{k \in [0,\infty[\mid f_k \text{ is differentiable}\}$. Then $A =$?

Answer:

2.4 What is the least value of $K > 0$ such that

$$|\sin^2 x - \sin^2 y| \leq K|x - y|$$

for all real numbers x and y?

Answer:

2.5 Let Γ be the circle in the complex plane with centre at $z = 1$ and of radius unity. Evaluate:

$$\int_\Gamma \frac{z\, dz}{(z-1)^4}.$$

Answer:

2.6 If the plane $\mathbb{R}^2$ is provided with the Lebesgue measure, what is the measure of the set

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}?$$

Answer:

2.7 Pick out the sequences which are uniformly convergent:
(a) $f_n(x) = \sin^n x$ on $[0, \pi/2[$.
(b) $f_n(x) = \frac{x^n}{n} + 1$ on $[0, 1[$.
(c) $f_n(x) = \frac{1}{1+(x-n)^2}$ on $]-\infty, 0[$.
(d) $f_n(x) = \frac{1}{1+(x-n)^2}$ on $]0, +\infty[$.

Answer:

2.8 Pick out the functions which are Riemann integrable on the interval $[0, 1]$:
(a)

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

(b)

$$f(x) = \begin{cases} 1, & \text{if } x \in \{\alpha_1, \alpha_2, ..., \alpha_n\} \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha_1, ..., \alpha_n$ are fixed, but arbitrarily chosen numbers in $[0, 1]$.
(c)

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational or if } x = 0 \\ \sin q\pi, & \text{if } x = p/q,\ p \text{ and } q \text{ positive and coprime integers.} \end{cases}$$

Answer:

2.9 Pick out the functions from the following list which are analytic in $\mathbb{C}$:
(a) $f(z) = |z|^2$
(b) $f(z) = \overline{z}$
(c) $f(z) = \mathrm{Re(z)}$

Answer:

2.10 Pick out the statements which are true:
(a) $|\sin z| \leq 1$ for all $z \in \mathbb{C}$.
(b) $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.
(c) $\sin z = (e^{iz} - e^{-iz})/2$ for all $z \in \mathbb{C}$.

Answer:

Section 3: Topology

3.1 Let $f :]0,1[\to \mathbb{R}$ be continuous. It can be extended to a continuous function $\widetilde{f} : [0,1] \to \mathbb{R}$ if, and only if, it is ....

Answer:

3.2 Consider the disjoint closed sets in $\mathbb{R}^2$ given by

$$A \;=\; \{(x,y) \in \mathbb{R}^2 \mid y = 0\} \; and \; B \;= \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}.$$

What is the distance $d(A, B)$ between them?

Answer:

In Questions 3.3 and 3.4 below, write '0' if the set A is empty, the exact number of elements in it if the set is finite, and 'infinite' if the set is infinite.

3.3 Let $f : [0,1] \to [0,1]$ be such that $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$ for all $x, y \in [0,1]$. Let $A = \{x \in [0,1] \mid f(x) = x\}$. The number of elements in A is ....

Answer:

3.4 Let $f : [0,1] \to [0,1]$ be continuous and such that $f(0) = f(1)$. Let

$$A \;=\; \{(t,s) \in [0,1] \times [0,1] \mid t \neq s \;\text{ and }\; f(t) = f(s)\}.$$

The number of elements in A is ....

Answer:

In Questions 3.5 to 3.10 below, mark a **tick** over the topological properties true for the set A and **strike out** the properties that do not hold. Your answer will be treated as correct only if **all** the choices are correctly made.

3.5 Identify the space of all $n \times n$ matrices (with real entries) with $\mathbb{R}^{n^2}$. Let A be the set of all invertible matrices.

Answer: open, closed, connected, dense.

3.6 $A = f(B) \subset X$ where $B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2\}$, X is an arbitrary topological space and $f : \mathbb{R}^2 \to X$ is an arbitrary continuous map.

Answer: open, closed, compact, connected.

3.7 $A = X \backslash \{x_o\}$ where X is an arbitrary Hausdorff topological space and $x_o \in X$.

Answer: open, closed, connected, dense.

3.8 $A = f(B) \subset \mathbb{R}$ where B is a closed interval contained in $]0, \infty[$ and $f(t) = \log t$.

Answer: open, closed, connected, compact.

3.9 $A = \{(x, y) \in \mathbb{R}^2 \mid y = mx\} \backslash \{(0, 0)\} \subset \mathbb{R}^2$.

Answer: open, closed, connected, nowhere dense.

3.10 A is the closure in $\mathcal{C}[0, 1]$ of the set B where

$$B = \{f \in \mathcal{C}^1[0, 1] \mid |f(x)| \leq 1 \text{ and } |f'(x)| \leq 1 \text{ for all } x \in [0, 1]\}.$$

Answer: closed, compact, connected, dense.

Section 4: Applied Mathematics

4.1 Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $\mathbf{v} = (v_1, v_2, v_3)$ be a solenoidal vector field on $\mathbb{R}^3$. Evaluate:

$$\int_S [x(x + v_1(x, y, z)) + y(y + v_2(x, y, z)) + z(z + v_3(x, y, z))] dS.$$

Answer:

4.2 Let $\mathbf{v} = (v_1, v_2, v_3)$ be a vector field on $\mathbb{R}^3$ where $v_1 = \sqrt{1 + x^2 + y^2}$, $v_2 = \sqrt{1 + z^2}$ and $v_3 = \sqrt{1 + x^2 y^2 z^2}$. Evaluate $\mathrm{div}(\mathbf{curl}\ \mathbf{v})$.

Answer:

4.3 What is the smallest value of $\lambda \in \mathbb{R}$ such that the boundary value problem:

$$u''(x) + \lambda u(x) = 0 \text{ in }]0, 1[\text{ and } u(0) = u(1) = 0$$

has a non-trivial solution ($i.e.$ $u \not\equiv 0$)?

Answer:

4.4 Let $\mathbf{u}(t) = (u_1(t), u_2(t))$ be the unique solution of the problem:

$$\begin{aligned} \tfrac{d\mathbf{u}}{dt}(t) &= A\mathbf{u}(t), \ t > 0 \\ \mathbf{u}(0) &= \mathbf{u}_o \end{aligned}$$

where $\mathbf{u}_o = (1, 1)$ and A is a symmetric 2×2 matrix such that $\mathrm{tr}(A) < 0$ and $\det(A) > 0$. Evaluate:

$$\lim_{t \to \infty} u_1(t).$$

Answer:

4.5 Simpson's rule gives the exact value of $\int_0^1 p(t)dt$ for every polynomial of degree less than or equal to ....

Answer:

4.6 Consider the linear programming problem: Maximize $z = 5x + 7y$ such that

$$\begin{aligned} x - y &\leq 1 \\ 2x + y &\geq 2 \\ x + 2y &\leq 4 \\ x \geq 0,\ y &\geq 0. \end{aligned}$$

What is the optimal value of z?

Answer:

4.7 According to the classification of second order linear partial differential operators, the operator

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2}$$

is of .... type.

Answer:

4.8 Evaluate:

$$\int\int_{\mathbb{R}^2} e^{-(x+2y)^2-(x+y)^2} dx dy.$$

Answer:

4.9 A necessary and sufficient condition that the boundary value problem:

$$\begin{aligned} \tfrac{\partial^2 u}{\partial x^2} + \tfrac{\partial^2 u}{\partial y^2} &= f(x, y) \text{ in } \Omega \\ \tfrac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega \end{aligned}$$

(where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of the function u) has a solution is ....

Answer:

4.10 The radius r and height h of a right circular cylinder of fixed volume V and least total surface area are connected by the relation ....

Answer:

Section 5: Miscellaneous

5.1 What is the maximum number of pieces that a pizza can be cut into by 7 knife strokes?

Answer:

5.2 Let n be a fixed positive integer. Let C_r denote the number of ways of choosing r objects from a collection of n objects. Evaluate:

$$C_1 + 2.\ C_2 + ... + n.\ C_n.$$

Answer:

5.3 Inside a square of side 2 units, five points are marked at random. What is the probability that there are at least two points such that the distance between them is at most $\sqrt{2}$ units?

Answer:

5.4 What is the area of the triangle in the complex plane formed by the points representing $1, \omega$ and ω^2, where ω is a complex cube root of unity?

Answer:

5.5 What is the number of points of intersection, in $\mathbb{R}^2$, of the two plane curves $(1 + x^2 + y^2)(x^2 + y^2 - 4) = 0$ and $y = 7x$?

Answer:

5.6 What geometric figure is formed by the locus of a point which moves so that the sum of four times its distance from the x-axis and nine times its distance from the y-axis is equal to 10?

Answer:

5.7 A real number is *algebraic* if it is the root of a polynomial with integer coefficients. Define $A : [0,1] \to \mathbb{R}$ by

$$A(x) \;=\; \begin{cases} 1 & \text{if } x \text{ is algebraic} \\ 0 & \text{otherwise} \end{cases}$$

Evaluate: $\int_0^1 A(x)dx$.

Answer:

5.8 In the rectangle $[0, \pi/2] \times [0,1] \subset \mathbb{R}^2$, a point (x, y) is chosen at random. What is the probability that $y \leq \sin x$?

Answer:

5.9 If p is a prime greater than, or equal to, 11, then, either $p^3 - 1$ or $p^3 + 1$ is divisible by 14. True or False?

Answer:

5.10 Evaluate:

$$\sum_{n=1}^{\infty} \frac{n^2 - n + 1}{n!}.$$

Answer: