Quasi optimal adaptive pseudostress approximation of the Stokes equations

joint work with Carsten Carstensen and Mira Schedensack

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The pseudostress-velocity formulation [3, 5] of the stationary Stokes equations

$$-\Delta u + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega \tag{1}$$

with Dirichlet boundary conditions along the polygonal boundary $\partial\Omega$ allows the stresses-like variables σ in Raviart-Thomas mixed finite element spaces [2] $\mathrm{RT}_k(\mathcal{T})$ with respect to a regular triangulation $\mathcal{T}$, and hence allows for higher flexibility in arbitrary polynomial degrees.

The weak form of problem (1) is formally equivalent and reads: Given $f \in L^2(\Omega;\mathbb{R}^2)$ and $g \in H^1(\Omega;\mathbb{R}^2) \cap H^1(\mathcal{E}(\partial\Omega);\mathbb{R}^2)$ with $\int_{\partial\Omega} g\cdot\nu\,ds = 0$ seek $\sigma \in H(\operatorname{div},\Omega;\mathbb{R}^{2\times 2})/\mathbb{R}$ and $u \in L^2(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned} \int_\Omega \operatorname{dev}\sigma : \tau\,dx + \int_\Omega u\cdot\operatorname{div}\tau\,dx &= \int_{\partial\Omega} g\cdot\tau\,\nu\,ds,\\ \int_\Omega v\cdot\operatorname{div}\sigma\,dx &= -\int_\Omega f\cdot v\,dx \end{aligned} \tag{2}$$

for all $(\tau, v) \in H(\operatorname{div},\Omega;\mathbb{R}^{2\times 2})/\mathbb{R} \times L^2(\Omega;\mathbb{R}^2)$, where the deviatoric part of the tensor σ reads $\operatorname{dev}\sigma := \sigma - 1/2\ \mathrm{tr}(\sigma) I_{2\times 2}$. The discrete formulation of (2) seeks $\sigma_{\mathrm{PS}} \in \mathrm{PS}(\mathcal{T}) := \mathrm{RT}_k(\mathcal{T}) \cap H(\operatorname{div},\Omega;\mathbb{R}^{2\times 2})/\mathbb{R}$ and $u_{\mathrm{PS}} \in P_0(\mathcal{T}_\ell;\mathbb{R}^2)$ such that

$$\begin{aligned} \int_\Omega \operatorname{dev}\sigma_{\mathrm{PS}} : \tau_{\mathrm{PS}}\,dx + \int_\Omega \operatorname{div}\tau_{\mathrm{PS}}\cdot u_{\mathrm{PS}}\,dx &= \int_{\partial\Omega} g\cdot\tau_{\mathrm{PS}}\nu\,ds\\ \int_\Omega \operatorname{div}\sigma_{\mathrm{PS}}\cdot v_{\mathrm{PS}}\,dx &= -\int_\Omega f\cdot v_{\mathrm{PS}}\,dx \end{aligned}$$

for all $(\tau_{\mathrm{PS}}, v_{\mathrm{PS}}) \in \mathrm{PS}(\mathcal{T}) \times P_k(\mathcal{T};\mathbb{R}^2)$.

The reliability and efficiency up to data oscillations of the explicit residual-based error estimator η_ℓ have been established in [5]. The contributions on each triangle T with edges $E \in \mathcal{E}(T)$ and tangents τ_E and jumps $[\cdot]_E$ read

$$\begin{aligned} \eta^2(T) := \mathrm{osc}^2(f,T) + |T|\,\|\mathrm{curl}(\operatorname{dev}\sigma_{\mathrm{PS}})\|^2_{L^2(T)}\\ + |T|^{1/2}\sum_{E\in\mathcal{E}(T)} \|[\operatorname{dev}\sigma_{\mathrm{PS}}]_E\tau_E\|^2_{L^2(E)}. \end{aligned}$$

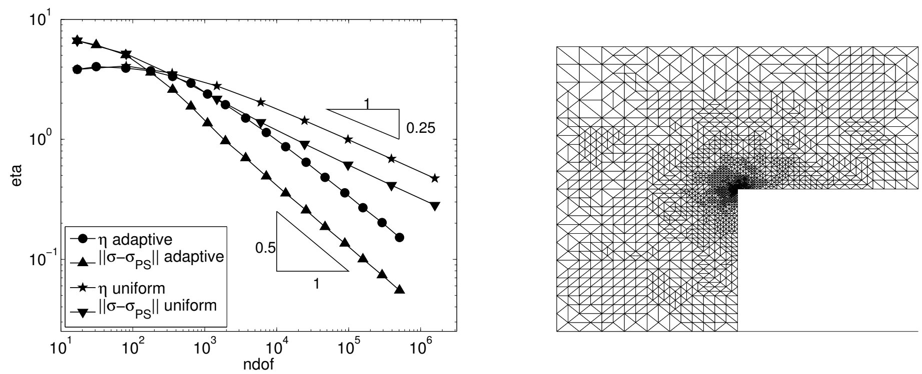

Figure 1: Convergence history for uniform and adaptive mesh-refinement in the L-shaped domain example and mesh generated by Apsfem.

This gives rise to run the following adaptive algorithm Apsfem with the steps Solve, Estimate, Mark, Refine, in each loop iteration.

Input: Initial triangulation $\mathcal{T}_0$, bulk parameter $0 < \theta < \theta_0 \ll 1$

Loop: For $\ell = 0, 1, 2, \ldots$

Solve. Compute (u_ℓ, σ_ℓ) with respect to the triangulation $\mathcal{T}_\ell$

Estimate. Compute the piecewise contributions of η_ℓ

Mark. Mark minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ such that

$$\theta \eta_\ell^2 \le \eta_\ell^2(\mathcal{M}_\ell) := \sum_{T \in \mathcal{M}_\ell} \eta_\ell^2(T).$$

Refine. Refine $\mathcal{M}_\ell$ in $\mathcal{T}_\ell$ with newest vertex bisection, generate $\mathcal{T}_{\ell+1}$

Output: Sequences $(\mathcal{T}_\ell)_\ell$ and $(u_\ell, \sigma_\ell)_\ell$

The definition of quasi-optimal convergence is based on the concept of approximation classes. For $s > 0$, let

$$\mathcal{A}_s := \big\{(\sigma, f, g) \in H(\mathrm{div}, \Omega; \mathbb{R}^{2\times 2})/\mathbb{R} \times L^2(\Omega; \mathbb{R}^2) \times \big(H^1(\Omega; \mathbb{R}^2) \cap H^1(\mathcal{E}(\partial\Omega); \mathbb{R}^2)\big) \,\big|\, |(\sigma, f, g)|_{\mathcal{A}_s} < \infty\big\}$$

with $|(\sigma, f, g)|_{\mathcal{A}_s} :=$

$$\sup_{N \in \mathbb{N}} N^s \inf_{|\mathcal{T}| - |\mathcal{T}_0| \le N} \Big(\|\mathrm{dev}(\sigma - \sigma_{\mathcal{T}})\|^2_{L^2(\Omega)} + \mathrm{osc}^2(f, \mathcal{T}) + \mathrm{osc}^2\big(\frac{\partial g}{\partial s}, \mathcal{E}(\partial\Omega)\big)\Big)^{1/2}.$$

In the infimum, $\mathcal{T}$ runs through all admissible triangulations (with respective discrete solutions $\sigma_{\mathcal{T}}$) that are refined from $\mathcal{T}_0$ by newest vertex bisection of

[1, 7] and that satisfy $|\mathcal{T}| - |\mathcal{T}_0| \leq N$. The main result relies on a novel observation from ongoing work of Carstensen, Kim, and Park on the equivalence with nonconforming schemes in the spirit of [6] and is therefore restricted to the lowest-order Raviart-Thomas finite element functions. The main theorem states quasi-optimal convergence in the following sense. For any sufficienty small bulk parameter $0 < \theta < \theta_0$ and $(\sigma, f, g) \in \mathcal{A}_s$, APSFEM generates sequences of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions $(u_\ell, \sigma_\ell)_\ell$ of optimal rate of convergence in the sense that

$$(|\mathcal{T}_\ell| - |\mathcal{T}_0|)^s \Big(\|\mathrm{dev}(\sigma - \sigma_\ell)\|^2_{L^2(\Omega)} + \mathrm{osc}^2(f, \mathcal{T}_\ell) + \mathrm{osc}^2(\partial g/\partial s, \mathcal{E}_\ell(\partial\Omega)) \Big)^{1/2} \\ \leq C_{\mathrm{opt}} \, |(\sigma, f, g)|_{\mathcal{A}_s} .$$

The main ingredients of the proof are the quasi-orthogonality, which leads to a contraction of some linear combination of error, estimator, and data oscillations, and the discrete reliability. Those are established for the lowest-order case.

References

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