# Comparison results for first-order FEMs

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Various first-order finite element methods are known for the Poisson Model Problem (1) and for linear elasticity (2). The recent publications [1] started the comparison between some of these methods for the Poisson Model Problem, which is completed in this presentation and its underlying paper [3].

Given a bounded polygonal Lipschitz domain $\Omega$ in the plane and data $f \in L^2(\Omega)$, the Poisson model problem seeks the weak solution $u \in H^1(\Omega)$ of

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

This presentation compares the error of three popular finite element methods (FEM) of Figure 1 for the numerical solution of (1), namely the conforming *Courant* FEM (CFEM) [4], the nonconforming *Crouzeix-Raviart* FEM (CR-NCFEM) [5], and the mixed *Raviart-Thomas* FEM (RT-MFEM) [7] with respective solutions $u_{\mathrm{C}}$, $u_{\mathrm{CR}}$, and $(p_{\mathrm{RT}}, u_{\mathrm{RT}})$ based on a shape-regular triangulation $\mathcal{T}$ of $\Omega$ into triangles. The finite element space of CFEM reads $P_1(\Omega) \cap C_0(\Omega)$

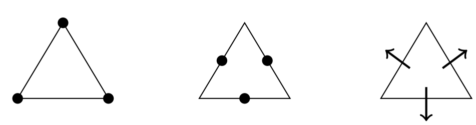

Figure 1: CFEM (left), CR-NCFEM (middle), RT-MFEM (right).

for $C_0(\Omega)$ the continuous functions with zero boundary conditions. The finite element space of Crouzeix-Raviart $\mathrm{CR}_0^1(\mathcal{T})$ consists of all piecewise affines which are continuous at the midpoints of interior edges and vanish at the midpoints of exterior edges. The Raviart-Thomas finite element space for the flux approximation reads $\mathrm{RT}_0(\mathcal{T}) := \{p_{\mathrm{RT}} \in P_1(\mathcal{T}, \mathbb{R}^2) \cap H(\mathrm{div}, \Omega) \,|\, \forall T \in \mathcal{T}\, \exists\, a_T, b_T, c_T \in \mathbb{R} : p_{\mathrm{RT}}|_T(x) = (a_T, b_T) + c_T x\}$.

The comparison is stated in terms of $A \lesssim B$ which abbreviates the existence of some constant $C$ which only depends on the minimal angle in $\mathcal{T}$, but not on the domain $\Omega$ and not on the mesh-size $h_{\mathcal{T}}$, such that $A \leq CB$. The comparison includes data oscillations, namely $\mathrm{osc}(f, \mathcal{T}) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|$ where $\Pi_0$ denotes the $L^2$ orthogonal projection onto the piecewise constants.

The comparison result for CFEM, CR-NCFEM and RT-MFEM states that the errors of CFEM and CR-NCFEM are equivalent up to data oscillations, in the sense that

$$\|\nabla u - \nabla u_{\mathrm{C}}\| \lesssim \|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| \lesssim \|\nabla u - \nabla u_{\mathrm{C}}\| + \operatorname{osc}(f, \mathcal{T}).$$

The error of RT-MFEM is superior in the sense that

$$\|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| \lesssim \|h_{\mathcal{T}} f\| + \|\nabla u - p_{\mathrm{RT}}\| \lesssim \|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| + \operatorname{osc}(f, \mathcal{T}),$$

but the converse is false, i.e.,

$$\|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| \not\lesssim \|\nabla u - p_{\mathrm{RT}}\| + \operatorname{osc}(f, \mathcal{T}).$$

The proof of the inequalities is an example of the medius analysis for it combines arguments of an *a priori* with those of an *a posteriori* error analysis. It is emphasised that no regularity assumption is made and the results hold for arbitrary coarse triangulations and not just in an asymptotic regime. The proof of the superiority of RT-MFEM considers a sequence of domains, on which the RT-MFEM has a steeper convergence rate than CR-NCFEM.

The results for the Poisson Model Problem can be generalised for the Navier-Lamé equations from linear elasticity, which seek $u \in H_0^1(\Omega; \mathbb{R}^2)$ with

$$f + 2\mu\Delta u + (\mu + \lambda)\nabla(\operatorname{div} u) = 0 \quad \text{in } \Omega. \tag{2}$$

The compared FEMs are the conforming *Courant* FEM (CFEM) [2], the nonconforming *Kouhia-Stenberg* FEM (KS-NCFEM) [6], and the nonconforming *Crouzeix-Raviart* FEM (CR-NCFEM) [2] with respective solutions $\sigma_{\mathrm{C}}$, $\sigma_{\mathrm{KS}}$ and $\sigma_{\mathrm{CR}}$. The finite element space of KS-NCFEM reads $\mathrm{KS} := (P_1(\mathcal{T}) \cap C_0(\Omega)) \times \mathrm{CR}_0^1(\mathcal{T})$. The discretisation of CFEM and KS-NCFEM is based on the bilinear form

$$a(u_{\mathrm{KS}}, v_{\mathrm{KS}}) := \int_\Omega \varepsilon_{\mathrm{NC}}(u_{\mathrm{KS}}) : \mathbb{C}\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\, dx,$$

while the discretisation of CR-NCFEM involves the bilinear form

$$a(u_{\mathrm{CR}}, v_{\mathrm{CR}}) := \int_\Omega \big(\mu\, D_{\mathrm{NC}} u_{\mathrm{CR}} : D_{\mathrm{NC}} v_{\mathrm{CR}} + (\mu + \lambda)\, \operatorname{div}_{\mathrm{NC}} u_{\mathrm{CR}} \operatorname{div}_{\mathrm{NC}} v_{\mathrm{CR}}\big) dx.$$

The comparison result for linear elasticity involves the Lamé modulus $\lambda$, which effects the locking and the $\lesssim$ notation means, that, in addition, the underlying constants do not depend on the Lamé modulus $\lambda$. Then

$$\|\sigma - \sigma_{\mathrm{C}}\| \lesssim \lambda\, \|\sigma - \sigma_{\mathrm{KS}}\| \lesssim \lambda\, \big(\|\sigma - \sigma_{\mathrm{C}}\| + \operatorname{osc}(f, \mathcal{T})\big)$$

and

$$\|\sigma - \sigma_{\mathrm{KS}}\| + \operatorname{osc}(f, \mathcal{T}) \approx \|\sigma - \sigma_{\mathrm{CR}}\| + \operatorname{osc}(f, \mathcal{T}).$$

## References


[1] D. Braess, *An a posteriori error estimate and a comparison theorem for the nonconforming $P_1$ elemen*, Calcolo **46** (2009), 149–155.

[2] S.C. Brenner, L.R. Scott, *The mathematical theory of finite element methods*, Texts in Applied Mathematics **15** (2008).

[3] C. Carstensen, D. Peterseim, M. Schedensack, *Comparison Results of Three First-order Finite Element Methods for the Poisson Model Problem*, Preprint **#831** (2011).

[4] R. Courant, *On a method for the solution of boundary-value problems*, in: Theodore von Kármán Anniversary Volume (1941), 189–194.

[5] M. Crouzeix, P.A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **7** (1973), 33–75.

[6] R. Kouhia, R. Stenberg, *A linear nonconforming finite element method for nearly incompressible elasticity and Stokes flow*, Comput. Methods Appl. Mech. Engrg. **124** (1995), 195–212.

[7] P.A. Raviart, J.M. Thomas *A mixed finite element method for 2nd order elliptic problems*, in: Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975) (1977), 292–315.