A CRITERION FOR A DEGREE-ONE HOLOMORPHIC MAP TO BE
A BIHOLOMORPHISM

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Abstract. Let \(X\) and \(Y\) be compact connected complex manifolds of the same
dimension with \(b_2(X) = b_2(Y)\). We prove that any surjective holomorphic map of degree one
from \(X\) to \(Y\) is a biholomorphism. A version of this was established by the first two
authors, but under an extra assumption that \(\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)\). We
show that this condition is actually automatically satisfied.

1. Introduction

Let \(X\) and \(Y\) be compact connected complex manifolds of dimension \(n\). Let
\(f : X \to Y\)
be a surjective holomorphic map such that the degree of \(f\) is one, meaning that the
pullback homomorphism
\[
\mathbb{Z} \cong H^{2n}(Y, \mathbb{Z}) \xrightarrow{f^*} H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}
\]
is the identity map of \(\mathbb{Z}\). It is very natural to ask, “Under what conditions would \(f\) be a
biholomorphism?” An answer to this was given by [2, Theorem 1.1], namely:

**Result 1** ([2, Theorem 1.1]). Let \(X\) and \(Y\) be compact connected complex manifolds of
dimension \(n\), and let \(f : X \to Y\) be a surjective holomorphic map such that the degree
of \(f\) is one. Assume that

*(i)* the \(C^\infty\) manifolds underlying \(X\) and \(Y\) are diffeomorphic, and

*(ii)* \(\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)\).

Then, the map \(f\) is a biholomorphism.

In the proof of Result 1, the condition *(i)* is used only in concluding that \(\dim H^2(X, \mathbb{R}) = \dim H^2(Y, \mathbb{R})\). In other words, the proof of [2, Theorem 1.1] establishes that if

\[
\dim H^2(X, \mathbb{R}) = \dim H^2(Y, \mathbb{R}) \quad \text{and} \quad \dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y),
\]

then — with \(X, Y\), and \(f\) as above — \(f\) is a biholomorphism.

There is some cause to believe that the condition *(ii)* in Result 1 might be superfluous
(which we shall discuss presently). It is the basis for our main theorem, which gives a
simple, purely topological, criterion for a degree-one map to be a biholomorphism:

**Theorem 2.** Let \(X\) and \(Y\) be compact connected complex manifolds of dimension \(n\),
and let \(f : X \to Y\) be a surjective holomorphic map of degree one. Then, \(f\) is a
biholomorphism if and only if the second Betti numbers of \(X\) and \(Y\) coincide.

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If $X$ and $Y$ were assumed to be Kähler, then Theorem 2 would follow from Result 1. This is because, by the Hodge decomposition, $\dim H^1(M, \mathcal{O}_M) = \frac{1}{2} \dim H^2(M, \mathbb{C})$ for any compact Kähler manifold $M$. We shall show that this observation — i.e., that condition (ii) in Result 1 is automatically satisfied under the hypotheses therein — holds true in the general, analytic setting. In more precise terms, we have:

**Proposition 3.** Let the manifolds $X$ and $Y$ and $f : X \to Y$ be as in Result 1. Then, $f$ induces an isomorphism between $H^1(X, \mathcal{O}_X)$ and $H^1(Y, \mathcal{O}_Y)$. In particular, $\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)$.

The above proposition might be unsurprising to many. It is well known when $X$ and $Y$ are projective. Since we could not find an explicit statement of Proposition 3 — and since certain supplementary details are required in the analytic case — we provide a proof of it in Section 2. The non-trivial step in proving Theorem 2 uses Result 1: given Proposition 3, our theorem follows from Result 1 and the comment above upon its proof.

### 2. Proof of Proposition 3

We begin with a general fact that we shall use several times below. For any proper holomorphic map $F : V \to W$ between complex manifolds, the Leray spectral sequence gives the following exact sequence:

$$0 \to H^1(W, F_\ast \mathcal{O}_V) \xrightarrow{\theta_F} H^1(V, \mathcal{O}_V) \to H^0(W, R^1F_\ast \mathcal{O}_V) \to \cdots. \quad (2.1)$$

With our assumptions on $X$, $Y$ and $f$, the map $f^{-1}$ (which is defined outside the image in $Y$ of the set of points at which $f$ fails to be a local biholomorphism) is holomorphic on its domain. Thus $f$ is bimeromorphic.

We note that any bimeromorphic holomorphic map of connected complex manifolds has connected fibers, because it is biholomorphic on the complement of a thin analytic subset. In particular, the fibers of $f$ are connected.

**Claim 1.** Let $F : V \to W$ be a bimeromorphic holomorphic map between compact, connected complex manifolds. The natural homomorphism

$$\mathcal{O}_W \to F_\ast \mathcal{O}_V \quad (2.2)$$

is an isomorphism.

By definition, (2.2) is injective. In our case, it is an isomorphism outside a closed complex analytic subset of $W$, say $\mathcal{S}$, of codimension at least 2. So, to show that (2.2) is surjective, it suffices to show that given any $w \in \mathcal{S}$, for each open connected set $U \ni w$ and each holomorphic function $\psi$ on $F^{-1}(U)$ there is a function $H_\psi$ holomorphic on $U$ such that

$$\psi = H_\psi \circ F \text{ on } F^{-1}(U).$$

Since $F^{-1}$ is holomorphic on $W \setminus \mathcal{S}$, we set

$$H_\psi|_{U \setminus \mathcal{S}} := \psi \circ (F^{-1}|_{U \setminus \mathcal{S}}).$$

This has a unique holomorphic extension to $U$ by Hartogs’ theorem (or more accurately: Riemann’s second extension theorem), since $\mathcal{S}$ is of codimension at least 2. As $F$ has compact, connected fibers, this extension has the desired properties. This shows that the homomorphism in (2.2) is surjective. Hence the claim.
By Claim 1, (2.1) yields an injective homomorphism
\[ \Theta_f : H^1(Y, \mathcal{O}_Y) \rightarrow H^1(X, \mathcal{O}_X), \] (2.3)
which is the composition of the homomorphism \( \theta_f \), as given by (2.1), and the isomorphism induced by (2.2).

There is a commutative diagram of holomorphic maps

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow h & & \downarrow f \\
Y & & Y
\end{array}
\]

(2.4)

where \( h \) is a composition of successive blow-ups with smooth centers, such that the subset of \( Y \) over which \( h \) fails to be a local biholomorphism (i.e., the image in \( Y \) of the the exceptional locus in \( Z \)) coincides with the subset of \( Y \) over which \( f \) fails to be a local biholomorphism. This fact (also called “Hironaka’s Chow Lemma”) can be deduced from Hironaka’s Flattening Theorem [4, p. 503], [4, p. 504, Corollary 1]. We recollect briefly the argument for this. The set \( \mathcal{A} \) of values of \( f \) in \( Y \) at which \( f \) is not flat coincides with the set of points over which \( f \) is not locally biholomorphic. Hironaka’s Flattening Theorem states that there exists a sequence of blow-ups of \( Y \) with smooth centers over \( \mathcal{A} \) amounting to a map
\[ h : Z \rightarrow Y \]
such that — with \( \tilde{Z} \) denoting the proper transform of \( Y \) in \( X \times_Y Z \) and \( \text{pr}_Z \) denoting the projection \( X \times_Y Z \rightarrow Z \) — the map \( \tilde{f} := \text{pr}|_{\tilde{Z}} \) is flat. In our case this implies that \( \tilde{f} : \tilde{Z} \rightarrow Z \) is a biholomorphism. The map \( g = \text{pr}_X \circ (\tilde{f})^{-1} \) and has the properties stated above.

The maps \( h \) and \( g \) above are proper modifications. Thus, all the assumptions in Claim 1 hold true for \( g : Z \rightarrow X \). Hence, we conclude that the homomorphism \( \mathcal{O}_X \rightarrow g_\ast \mathcal{O}_Z \) is an isomorphism. By (2.1) applied now to \((V, W, F) = (Z, X, g)\), the homomorphism
\[ \Theta_g : H^1(X, \mathcal{O}_X) \rightarrow H^1(Z, \mathcal{O}_Z), \] (2.5)
which is analogous to \( \Theta_f \) above, is injective.

Similarly, the homomorphism \( \mathcal{O}_Y \rightarrow h_\ast \mathcal{O}_Z \) is an isomorphism. Since (2.1), an exact sequence, is natural, we would be done — in view of (2.3), (2.5) and the diagram (2.4) — if we show that the homomorphism \( \Theta_h : H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Z, \mathcal{O}_Z) \), (given by applying (2.1) to \((V, W, F) = (Z, Y, h)\)) is an isomorphism.

To this end, we will use the following:

Claim 2. For a complex manifold \( W \) of dimension \( n \), if
\[ \sigma : S \rightarrow W \]
is a blow-up with smooth center, then the direct image \( R^1\sigma_\ast \mathcal{O}_S \) vanishes.

This claim is familiar to many. However, since it is not so easy to point to one specific work for a proof in the analytic case, we indicate an argument. We first study the blow-up \( \tilde{\sigma} : \tilde{S} \rightarrow \tilde{W} \) of a point \( 0 \in \tilde{W} \) with exceptional divisor \( \tilde{E} = \sigma^{-1}(0) \).

We use the “Theorem on formal functions” [3, Theorem 11.1], and the “Grauert comparison theorem” [1, Theorem III.3.1] for the analytic case. Let \( m_0 \subset \mathcal{O}_{\tilde{W}} \) be the maximal
ideal sheaf for the point $0 \in \widetilde{W}$. Then the completion $(R^1\sigma_*\mathcal{O}_{\widetilde{S}})_0$ of $(R^1\sigma_*\mathcal{O}_{\widetilde{S}})_0$ in the $m_0$-adic topology is equal to

$$\lim_{k} H^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}/\sigma^*(m_0^k)).$$

We have the exact sequence

$$0 \rightarrow \mathcal{O}_E(k) \rightarrow \mathcal{O}_{\widetilde{S}}/\sigma^*(m_0^{k+1}) \rightarrow \mathcal{O}_{\widetilde{S}}/\sigma^*(m_0^k) \rightarrow 0$$

of sheaves with support on

$$\sigma^{-1}(0) = \widetilde{E} \cong \mathbb{P}^{n-1}$$

so that the cohomology groups $H^q(\widetilde{E}, \mathcal{O}_{\widetilde{E}}(k))$ vanish for all $k \geq 0$, and $q > 0$. In particular the maps

$$H^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}/\sigma^*(m_0^{k+1})) \rightarrow H^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}/\sigma^*(m_0^k))$$

are isomorphisms for $k \geq 1$, and furthermore we have

$$H^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}/\sigma^*(m_0)) \cong H^1(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}) = 0.$$

This shows that $R^1\sigma_*\mathcal{O}_{\widetilde{S}}$ vanishes. This establishes the claim for blow-up at a point.

Now consider the case where the center of the blow-up $\sigma$ is a smooth submanifold $A$ of positive dimension. Since the claim is local with respect to the base space $W$, we may assume that $W$ is of the form $A \times \tilde{W}$, where both $A$ and $\tilde{W}$ are small open subsets of complex number spaces, e.g. polydisks. Denote by $\pi : W \rightarrow \tilde{W}$ the projection. We identify $A$ with $A \times \{0\} = \pi^{-1}(0) \subset W$ as a submanifold.

Note that the blow-up

$$\sigma : S \rightarrow W$$

of $W$ along $A$ is the fiber product $\tilde{S} \times_{\tilde{W}} W \rightarrow W$. The exceptional divisor $E$ of $\sigma$ can be identified with $A \times \tilde{E}$.

In the above argument we replace the maximal ideal sheaf $m_0$ by the vanishing ideal $I_A$ of $A$. Now $\sigma^*(I_{A}^k)/\sigma^*(I_{A}^{k+1}) \cong \mathcal{O}_E(k)$, and by [1, Theorem III.3.4] we have

$$R^1(\sigma|_E)_*\mathcal{O}_E \cong \pi^* R^1(\sigma|_E)_*\mathcal{O}_E = 0$$

so that the earlier argument can be applied. Hence the claim.

Now, let

$$Z = Z_N \xrightarrow{\tau_N} Z_{N-1} \xrightarrow{\tau_{N-1}} \cdots \xrightarrow{\tau_1} Z_1 \xrightarrow{\tau_1} Z_0 = Y$$

be the sequence of blow-ups that constitute $h : Z \rightarrow Y$. We have $\tau_j_*\mathcal{O}_{Z_j} \cong \mathcal{O}_{Z_{j-1}}$ and $R^1\tau_{j,*}\mathcal{O}_{Z_j} = 0$ for $1 \leq j \leq \tau_N$. Combining these with (2.1) yields a canonical injective homomorphism

$$H^1(Z_j, \mathcal{O}_{Z_j}) \rightarrow H^1(Z_{j-1}, \mathcal{O}_{Z_{j-1}})$$

that is an isomorphism for all $j = 1, \cdots, N$. Hence, by naturality, the homomorphism $\Theta_h : H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Z, \mathcal{O}_Z)$ is an isomorphism. By our above remarks, this establishes the result.

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