MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018
HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: OCTOBER 20, 2018**

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1. In this problem $T_{f,n}(x; a)$ denotes the Taylor polynomial of f of degree n around a.

i) Prove Taylor's Approximation Theorem:

Let I be an open interval and let $n \in \mathbb{Z}_+$. Let $f : I \longrightarrow \mathbb{R}$ and suppose $f^{(n)}$ exists on I. Fix $a \in I$. Then, for any $x \in I$, $x \neq a$,

$$f(x) - T_{f,n}(x; a) \textit{ is } o(|x - a|^n) \textit{ as } x \to a.$$

ii) Suppose we are not interested in the freedom to pick **any** convenient $a \in I$ to approximate $f(x)$ in the vicinity of such an a. Suppose I is as above and $f : I \longrightarrow \mathbb{R}$. Let $n \in \mathbb{Z}_+$. Suppose $a_* \in I$ is such that $f^{(n)}(a_*)$ exists (we have no information about f at other points). Is it true that

$$f(x) - T_{f,n}(x; a_*) \text{ is } o(|x - a_*|^n) \text{ as } x \to a_*?$$

2–5. Problems 9, 10, 26, and 27 from "Baby" Rudin, Chapter 5.

6. Let $a \in \mathbb{R}$. Let $f, g : [a, +\infty) \longrightarrow \mathbb{R}$ be two continuous functions with $f(a) = g(a)$. Suppose f and g are differentiable on $(a, +\infty)$ and $f' \geq g'$ on $(a, +\infty)$. Then how are f and g related?

7. Our study of Chapter 9 from "Baby" Rudin will presume that everyone in class is familiar with basic linear algebra. With thus presupposition, let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and write

$$\|A\| = \sup_{v \,:\, \|v\|=1} \|Av\|,$$

where the notation $\|w\|$, for any vector w in some Euclidean space, will denote its Euclidean norm. Study Theorems 9.7 and 9.8 from "Baby" Rudin. (**Caution:** our notation $\|\cdot\|$ for vectors appears as $|\cdot|$ in the book.)