MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018
HOMEWORK 11

Instructor: GAUTAM BHARALI **Assigned: NOVEMBER 3, 2018**

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1. Let I be an open interval and $f : I \longrightarrow \mathbb{R}$ be differentiable and injective on I. Then:

i) Show that $f(I)$ is an open interval.

ii) Suppose $y_0 \in f(I)$ is such that $f'\big(f^{-1}(y_0)\big) \neq 0$. Then, show that f^{-1} is differentiable at y_0 and that

$$(f^{-1})'(y_0) \;=\; \frac{1}{f'\big(f^{-1}(y_0)\big)}.$$

Tip. You may assume without proof that f^{-1} is continuous.

2. Define the function $f : \mathbb{R} \longrightarrow \{0,1\}$ as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $f|_{[a,b]} \notin \mathscr{R}([a,b])$ for any $a < b$.

3–5. Problems 7, 8, and 16 from "Baby" Rudin, Chapter 6.

6. Let $[a,b]$ be a compact interval and let $f, g \in \mathscr{R}([a,b])$. Let p and q be positive real numbers such that $p^{-1} + q^{-1} = 1$. Prove **Hölder's inequality:**

$$\left| \int_a^b fg(x)\,dx \right| \;\leq\; \left[\int_a^b |f(x)|^p dx \right]^{1/p} \left[\int_a^b |g(x)|^q dx \right]^{1/q},$$

by completing the outline provided by parts (a)–(c) of Problem 10 in "Baby" Rudin, Chapter 6.

7. Prove the following assertion:

Let $[a,b]$ be a compact interval and let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Let $\mathcal{D}_f$ denote the set of discontinuities of f. Suppose that, for each $n \in \mathbb{Z}_+$, there exists a pairwise disjoint collection of closed subintervals of $[a,b]$: $\mathscr{C}_n := \big\{I_1^{(n)}, \dots, I_{M(n)}^{(n)}\big\}$, such that

$$\mathcal{D}_f \subset \bigcup_{i=1}^{M(n)} I_j^{(n)} \quad \text{and} \quad \sum_{i=1}^{M(n)} \text{length}\,\big(I_j^{(n)}\big) \leq \frac{1}{n}$$

for each $n = 1, 2, 3, \dots$ Then $f \in \mathscr{R}([a,b])$.

by using the following hint:
Let $C > 0$ be such that $|f(x)| \leq C$ for each $x \in [a, b]$. Fix $\varepsilon > 0$. Show that there exists a partition on $[a, b]$

$$\mathbb{P} : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

for which we may split the set $\{1, \ldots, n\}$ into a disjoint union $G \cup B$ such that

$$\sum_{j \in G} (M_j - m_j) \Delta x_j < \varepsilon/2; \text{ and}$$

$$\sum_{j \in B} \Delta x_j < \frac{\varepsilon}{4C}.$$

Here M_j and m_j have the same meanings as introduced in class in defining the Riemann integral.