MATH 221 : ANALYSIS I – REAL ANALYSIS

AUTUMN 2018

HOMEWORK 13

Instructor: GAUTAM BHARALI **Assigned: NOVEMBER 24, 2018**

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1. Let S be a subset of $\mathbb{R}^n$, let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and let a be a limit point of S. Write $f = (f_1, \dots, f_m)$. Show that

$$\lim_{x \to a} f(x) = L = (\lambda_1, \dots, \lambda_m)$$

if and only if

$$\lim_{x \to a} f_j(x) = \lambda_j \text{ for each } j = 1, \dots, m.$$

Conclude from this that f is continuous if and only if $f_1, \dots, f_m$ are continuous.

Hint. It might be helpful — although there are other ways to prove the above — to use the fact that if $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, then

$$|x_j| \leq \|x\| \leq \sqrt{d} \sup_{1 \leq i \leq d} |x_i| \text{ for each } j = 1, \dots, d. \tag{1}$$

2–6. Problems 6, 8, 9, 11 and 13 from "Baby" Rudin, Chapter 9.

Remarks:

(a) Problem 6 illustrates — as discussed in class — why differentiability of a function f in n variables, $n \geq 2$, at a point $a \in \mathsf{dom}(f)$ is **not** defined in terms of the existence of all the (first-order) partial derivatives of f at a!

(b) Concerning Problem 9: it might be useful to remember the fact — mentioned in class but not frequently used — that if S is a subset of a connected metric space (X, d) such that S is both open and closed, then it cannot be a proper subset.

7. The second inequality in (1) is a special case of a more general phenomenon. Let $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ be two norms on $\mathbb{R}^d$. Show that there exist constants $c, C > 0$ such that

$$c\|x\|^{(2)} \leq \|x\|^{(1)} \leq C\|x\|^{(2)} \;\; \forall x \in \mathbb{R}^d. \tag{2}$$

Hint. Clearly, it suffices to establish (2) with $\|\cdot\|^{(1)}$ being some fixed norm — the Euclidean norm, say (which is convenient because we defined boundedness in $\mathbb{R}^d$ terms of this norm). Show that the set $\{x \in \mathbb{R}^d : \|x\|^{(2)} = 1\}$ is bounded reckoned by the fixed norm.