MATH 221 : ANALYSIS I – REAL ANALYSIS

AUTUMN 2018

HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: AUGUST 10, 2018**

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1. Recall the binary operations $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by Peano Arithmetic. Prove that both operations are commutative.

2. Let A be a non-zero natural number. We say that a natural number d *divides* A — denoted by $d|A$ — if there exists some natural number k such that $A = kd$. Let p be a prime number. **Assume** the following fact: if a and b are non-zero natural numbers and $p|ab$, then either $p|a$ or $p|b$.
Now show that the equation $x^2 = p$ has no rational solution.

3. Let S be a non-empty set, and let $\sim$ be an equivalence relation on S. Recall that, for any $s \in S$, the *equivalence class of* s — denoted by $[s]$ — is defined as

$$[s] := \{x \in S : x \sim s\}.$$

Prove that $\sim$ partitions S into disjoint equivalence classes.

4. Let $(S, <)$ be an ordered set and let E be a subset of S. If $\sup E$ exists, then show that it is unique.

5. Consider a set $A = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ on which we define two binary operations $+$ and $\cdot$ as follows:

$$\overline{a} + \overline{b} := \overline{(a+b) \bmod 6}, \qquad \overline{a} \cdot \overline{b} := \overline{(a\,b) \bmod 6}. \tag{1}$$

(Given any $a \in \mathbb{N}$, we define "$a \bmod 6$" as follows:

$a \bmod 6$
:= the unique remainder belonging to the set $\{0, 1, 2, \dots, 5\}$ obtained when dividing a by 6.)

The operations between the unbarred variables a and b in (1) are the usual addition and multiplication on $\mathbb{N}$. Is $(A, +, \cdot)$ a field? Justifiy your answer.

6. Write $\mathbb{F}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ on which we define the binary operations $+$ and $\cdot$ analogous to those in the previous problem:

$$\overline{a} + \overline{b} := \overline{(a+b) \bmod 3}, \qquad \overline{a} \cdot \overline{b} := \overline{(a\,b) \bmod 3}.$$

Show that $(\mathbb{F}_3, +, \cdot)$ is a field.

Remark. The above holds true with any prime number replacing 3. The non-trivial step, in that case, is to show that multiplicative inverses exist — which follows from a result called *Fermat's Little Theorem.*