MATH 221 : ANALYSIS I – REAL ANALYSIS

AUTUMN 2018

HOMEWORK 2

Instructor: GAUTAM BHARALI **Assigned: AUGUST 17, 2018**

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1. Let $\mathbb{C}$ denote the field of complex numbers (i.e., equipped with the standard addition and multiplication operations). Show that there **cannot** exist any order relation "$<$" on $\mathbb{C}$ that would make $(\mathbb{C}, <)$ an ordered field.

Hint. You may freely use **without proof** some of the consequences of being an ordered field given in the section entitled *Fields* in Chapter 1 of "Baby" Rudin.

2. Recall that if α is a positive cut, then we define

$$\alpha^{-1} := \{x \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ such that } r < 1/x \text{ and } r \notin \alpha\} \cup 0^* \cup \{0\}.$$

(a) Define α^{-1} for a negative cut.

(b) Show that α^{-1} as defined **is** a cut for any $\alpha \neq 0^*$.

3. Read the proof of the Cauchy–Schwarz inequality on $\mathbb{C}^n$ given in Chapter 1 of "Baby" Rudin. Using this proof, show the following statement:

For two vectors $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$ the Cauchy–Schwarz inequality involving $\boldsymbol{a}$ and $\boldsymbol{b}$ occurs as an equality if and only if one vector is a scalar multiple of the other.

4. For any vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, define $\|\boldsymbol{x}\| := \sqrt{\sum_{1 \leq j \leq n} x_j^2}$. Prove that

$$\big|\, \|\boldsymbol{x}\| - \|\boldsymbol{y}\| \,\big| \leq \|\boldsymbol{x} - \boldsymbol{y}\| \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

5. Let A be a non-empty at most countable set and suppose, for each $\alpha \in A$, we are given a set B_α that is at most countable. *We know that $S := \bigcup_{\alpha \in A} B_\alpha$ is at most countable.* Now suppose that A is *countable*, and assume that $B_\alpha \neq B_{\alpha'}$ for $\alpha \neq \alpha'$. Is S countable? If yes, then give a justification, else give a counterexample.