MATH 221 : ANALYSIS I – REAL ANALYSIS

AUTUMN 2018

HOMEWORK 3

Instructor: GAUTAM BHARALI **Assigned: AUGUST 25, 2018**

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1. Let S be a non-empty subset of $\mathbb{N}$. Show that S contains a unique least element (with respect to the standard order "$<$" on $\mathbb{R}$).

In the next two problems, given a set S, the *power set* of S — denoted by $\mathcal{P}(S)$ — will refer to the set of all subsets of S.

2. Let S be a non-empty set. Show that the power set of S has the same cardinality has the set of all functions from S to the set $\{0, 1\}$.

3. Let S be an uncountable set. Show that:

(a) There exists an injective function from S into $\mathcal{P}(S)$.

(b) S does **not** have the same cardinality $\mathcal{P}(S)$.

Hint. The conclusions of Problem 2 above would be helpful.

4. (Problem 2 from Rudin, Chapter 2) A complex number is said to be an *algebraic number* if it is the root of an algebraic equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

for some $n \in \mathbb{Z}_+$, where $a_1, \dots, a_n$ are integers.
Show that the set of all algebraic integers is countable.

5. A *graph* $G := G(V, E)$ is a pair of sets (V, E), where V is a non-empty, at most countable set, and $E \subset T(V)$, where

$$T(V) \;:=\; \{\{x, y\} : x, y \in V,\ x \neq y\}.$$

The set V is called the set of *vertices of* G, and E is called the set of *edges of* G. Consider the following definitions:

- Given $x \neq y \in V$, a *path joining* x *to* y is a finite collection of edges $\{\{x_j, y_j\} \in E : j = 0, \dots, N\}$ such that $x_0 = x$, $y_{j-1} = x_j$, $j = 1 \dots N$, and $y_N = y$. The *length* of a path is the number of edges contained in it.
- The graph $G(V, E)$ is said to be *connected* if, for each $x \neq y \in V$, there is at least one path joining x to y.
- If $G(V, E)$ is a connected graph, define the function $d : V \times V \longrightarrow [0, \infty)$ by

$$d(x, y) \;=\; \begin{cases} 0, & \text{if } x = y, \\ \min\{\text{length}(P) : P \textit{ is a path joining } x \textit{ to } y\}, & \text{if } x \neq y. \end{cases}$$

Given any connected graph $G = G(V, E)$, is (V, d) a metric space? If yes, then give justifications, else give a counterexample.