MATH 221 : ANALYSIS I – REAL ANALYSIS

AUTUMN 2018

HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: AUGUST 31, 2018**

---

1. Let (X, d) be a metric space and $\{S_\alpha : \alpha \in A\}$ an arbitrary collection of subsets of X. State whether the correct relation **in general** should be $B \supset C$ or $B \subset C$ or $B = C$, where

$$B = \bigcup_{\alpha \in A} \overline{S}_\alpha \quad \text{and} \quad C = \overline{\bigcup_{\alpha \in A} S_\alpha}.$$

If $B \neq C$ in general, then provide an example showing that the relevant inclusion could be a strict inclusion.

2. Given a metric space (X, d) and a set $S \subset X$, we say that a point $x \in S$ is an *interior point of* S if there exists an $r > 0$ such that $B(x; r) \subset S$. Show that $S^\circ =$ the set of all interior points of S.

3. Show that the set-theoretic density property of $\mathbb{Q} \subset \mathbb{R}$ is equivalent to saying that $\mathbb{Q}$ is dense in $\mathbb{R}$ endowed with the standard metric.

4. (Problem 24 from Rudin, Chapter 2) A metric space is called *separable* if it has a countable dense subset. Now suppose (X, d) is a metric space in which every infinite subset has a limit point. Show that X is separable.

5. In each of the following cases, determine using **only** the basic definition whether or not S is a compact subset of X (i.e., do **not** appeal to any theorems on compactness):

(*a*) $(X, d) = \mathbb{R}$ equipped with the standard metric; $S = \{1/n = n : 1, 2, 3, \dots\} \cup \{0\}$.

(*b*) $(X, d) =$ any set containing at least two points, equipped with the 0–1 metric; $S \subset X$ (your answer will depend on the nature of S; please give a **complete** discussion).

(*c*) $X =$ the set of all bounded sequences in $\mathbb{R}$;

$$d\big(\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}\big) := \sup_{n\in\mathbb{N}} |x_n - y_n|\,;$$

and $S = \big\{ \{x_n\}_{n\in\mathbb{N}} : x_n \in [-1, 1]\ \forall n \in \mathbb{N} \big\}$ (here, you will first need to show that the given d is a metric).