MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018
HOMEWORK 6

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1. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. Show that $\{x_n\}$ converges to a point $x_0 \in X$ if and only if every subsequence $\{x_{n_k}\}_{k \in \mathbb{Z}_+}$ converges to x_0.

2–5. Problems 9, 20–22 from Rudin, Chapter 3.

6. Fix a real number $a > 0$. Recall that if $p \in \mathbb{Q}$ and if $p = m/n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, then

$$a^p := (a^m)^{1/n}. \tag{1}$$

Given this, if now p is an arbitrary real number, then we define

$$a^p := \sup\{a^q : q \in \mathbb{Q} \text{ and } q \leq p\}. \tag{2}$$

Assuming, for the moment, that the definition (1) does not depend on the choice of the representative m/n of p, show that:

(a) for $p \in \mathbb{Q}$ the two definitions (1) and (2) of a^p agree.

(b) for any $x, y \in \mathbb{R}$, $a^x a^y = a^{x+y}$.

7. Let $\{a_n\}$ be a sequence of real numbers. Define

$$A_k := \inf\{a_k, a_{k+1}, a_{k+2}, \dots\},$$
$$B_k := \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}.$$

Show that

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} A_k, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{k\to\infty} B_k.$$

Note. A part of what you need to show is that the sequences $\{A_k\}$ and $\{B_k\}$ are convergent in the **extended** real number system.

8. Complete the following outline for a proof that the interval $[0, 1)$ is uncountable. Given a number $x \in [0, 1)$, let $\mathcal{I}_0(x) := [0, 1)$ and define the intervals

$$\mathcal{I}_{n+1}(x) := \begin{cases} \big[\inf \mathcal{I}_n(x), \mu_n(x)\big), & \text{if } x < \mu_n(x), \\ \big[\mu_n(x), \sup \mathcal{I}_n(x)\big), & \text{if } x \geq \mu_n(x), \end{cases}$$

for $n = 0, 1, 2, \dots,$ where $\mu_n(x) := \big(\inf \mathcal{I}_n(x) + \sup \mathcal{I}_n(x)\big)/2$: i.e., the midpoint of $\mathcal{I}_n(x)$. Let $\mathfrak{S}$ denote the set of all sequences in $\{0, 1\}$. We now define a function $F : [0, 1) \to \mathfrak{S}$ as follows: write $F(x) = \{s_n(x)\}$ where

$$s_n(x) := \begin{cases} 0 & \text{if } x < \mu_{n-1}(x), \\ 1, & \text{if } x \geq \mu_{n-1}(x), \end{cases}$$

for $n = 1, 2, 3, \dots$.

(a) Show that the series

$$\sum_{n=1}^{\infty} \frac{s_n(x)}{2^n}$$

converges, and that its sum is x. (**Remark.** This problem shows that the "binary representation" of x — i.e., the expression "$0.s_1(x)\, s_2(x)\, s_3(x) \ldots$", which is analogous to the common decimal expressions for real numbers — exists.)

(b) Show that F is **not** surjective (use the conclusion of (a) above).

(c) Show that $\mathfrak{S} \setminus \mathsf{range}(F)$ is countable.

(d) Use the conclusions of (a)–(c) to show that $[0, 1)$ is uncountable.

9. Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges.

Hint. The partial sums of the above series have a feature that allows you to use a known convergence theorem.