MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018
HOMEWORK 7

Instructor: GAUTAM BHARALI **Assigned: SEPTEMBER 29, 2018**

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1. Prove that the set $[0,1] \times [0,1] \subset \mathbb{R}^2$ is uncountable.
Hint. You would need to use a result from Homework 6.

2. In class, we had proved the following:
Theorem. *Let (X,d) be a metric space, $S \subset X$, a a limit point of S and $f,g : S \to \mathbb{C}$. Suppose*

$$\lim_{x\to a} f(x) = \alpha \quad \textit{and} \quad \lim_{x\to a} g(x) = \beta.$$

Then,

(*a*) $\lim_{x\to a}(f+g)(x) = \alpha + \beta$.

(*b*) $\lim_{x\to a}(fg)(x) = \alpha\beta$.

(*c*) *Suppose $\beta \neq 0$. Then, there exists an $r > 0$ such that $g(x) \neq 0$ for each $x \in S \cap B(a;r)$. Furthermore, $\lim_{x\to a}(f/g)(x)$ exists and*

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\alpha}{\beta}.$$

by appealing to the sequential definition of the limit of a function (and, thus, to theorems for convergent sequences). Redo these proofs using the "ε–δ definition" of the limit of a function.
Remark. These proofs will require more effort, which reveals the usefulness of having multiple equivalent definitions.

3. Any rational number x can be written uniquely as $x = m/n$, where $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, and such that there is no $d \in \mathbb{N} \setminus \{0,1\}$ dividing both m and n — with the understanding that we take $n = 1$ when $x = 0$. (You may use this fact **without proof.** See Homework 1 for what "d divides m (or n)" means.) Define $f : \mathbb{R} \to \mathbb{Q}$ as follows:

$$f(x) := \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/n, & \text{if } x \in \mathbb{Q}, \end{cases}$$

where n is uniquely associated to $x \in \mathbb{Q}$ as explained above. Show that f is continuous at each irrational point and discontinuous at each rational point.

4. Given a non-empty set S, we say that $f : S \to \mathbb{C}$ is **bounded** if there exists a constant $M > 0$ such that $|f(x)| < M$ for every $x \in S$.
Prove that a set $S \in \mathbb{R}^n$ is compact if and only if every complex-valued continuous function $f : S \to \mathbb{C}$ is bounded.