MATH 221 : ANALYSIS I – REAL ANALYSIS

AUTUMN 2018

HOMEWORK 9

Instructor: GAUTAM BHARALI **Assigned: OCTOBER 13, 2018**

1–2. Problems 23 and 24 from "Baby" Rudin, Chapter 4.

3. Consider the result:

Theorem. *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $S \subsetneq X$ be dense subset. Let $f : S \longrightarrow Y$ be a uniformly continuous function. Suppose (Y, d_Y) is complete. Then, there exists a unique continuous function $\widetilde{f} : X \longrightarrow Y$ that extends f.*

that was *partially* proved in class. Consider the function $\widetilde{f}$ constructed in that proof, consider $a \in (X \setminus S)$, and let $\{x_n\}$ be a sequence in $X \setminus \{a\}$ that converges to a. Complete the following outline to prove that $\widetilde{f}$ is continuous:

(a) Explain why it suffices to only consider sequences $\{x_n\}$ such that

$$\mathsf{range}(\{x_n\}) \bigcap (X \setminus S) \text{ is an infinite set.} \tag{1}$$

(b) Consider a sequence $\{x_n\}$ with the property (1). Construct an auxiliary sequence $\{y_n\} \subset S$ such that for each n for which $x_n \notin S$, y_n is "sufficiently close" to x_n — in an appropriate sense — and converges to a in such a way that you can use its behaviour, plus uniform continuity, to infer that $\{\widetilde{f}(x_n)\}$ is convergent.

(c) Deduce that $\{\widetilde{f}(x_n)\}$ converges to $\widetilde{f}(a)$.

(d) Now, complete the argument showing that $\widetilde{f}$ is continuous.

4. Consider the continuous function $f : (a, b) \longrightarrow \mathbb{R}$ and define the *diagonal* $\nabla \subset (a, b) \times (a, b)$ as:

$$\nabla := \{(x, y) \in (a, b) \times (a, b) : x = y\}.$$

Define $F : (a, b) \times (a, b) \setminus \nabla \longrightarrow \mathbb{R}$ by:

$$F(x, y) := \frac{f(x) - f(y)}{x - y}.$$

Find a necessary and sufficient condition for F to admit a limit at each point $p \in \nabla$.

5–8. Problems 4–7 from "Baby" Rudin, Chapter 5.

9. Let $r \in \mathbb{R}$ and let p be a positive real number. Consider the function $f : [-1, 1] \longrightarrow \mathbb{R}$ given by:

$$f(x) := \begin{cases} x^r \sin(1/x^p), & \text{if } 0 < x \leq 1, \\ 0, & \text{if } -1 \leq x \leq 0. \end{cases}$$

Find (i) a necessary and sufficient condition on (r, p) for f to be differentiable at 0; (ii) a necessary and sufficient condition on (r, p) for f to be differentiable at 0 **and** such that f' is continuous at 0.