MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018

QUIZ 1 **AUGUST 20, 2018**

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1. Let S be a non-empty set, and let $\sim$ be an equivalence relation on S. Recall that, for any $s \in S$, the *equivalence class of* s — denoted by $[s]$ — is defined as

$$[s] := \{x \in S : x \sim s\}.$$

Prove that $\sim$ partitions S into disjoint equivalence classes.

Solution. Consider an arbitrary $s \in S$. Then $s \in [s]$ because $\sim$ is reflexive. Since s was chosen arbitrarily, this shows that

$$S = \bigcup_{s \in S} [s].$$

Thus, it suffices to show that if $s, t \in S$ and $s \neq t$, then:

$$\text{either } [s] = [t] \text{ or } [s] \bigcap [t] = \emptyset.$$

So, pick arbitrary $s, t \in S$. If $[s] \cap [t] = \emptyset$, there is nothing to show. Hence consider the case $[s] \cap [t] \neq \emptyset$. Thus $\exists x \in S$ such that $x \in [s]$ and $x \in [t]$. By definition:

$$x \sim \quad \text{and} \quad x \sim t. \tag{1}$$

Now consider any $y \in [s]$. Then, we have, by definition $y \sim s$. From (1) and by symmetry, we have

$$y \sim s \quad \text{and} \quad s \sim x.$$

This gives us $y \sim x$ by transitivity. Then by (1) and transitivity again, we have $y \sim t$. Thus $y \in [t]$, and as y was arbitrary, we have $[s] \subseteq [t]$. By symmetry, we can interchange the roles of s and t above to get $[t] \subseteq [s]$. So $[s] = [t]$, whence by our above remarks, we are done. ☐