MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018

QUIZ 2 **SEPTEMBER 3, 2018**

---

1. Let S be a non-empty subset of $\mathbb{N}$. Show that S contains a unique least element (with respect to the standard order "$<$" on $\mathbb{R}$).

Solution. There are **several** different approaches to solving the above problem. We present below one of these approaches.

Let S be a non-empty subset of $\mathbb{N}$. First, we shall show that if S has a least element, then it is unique. If possible, let there be two elements $a, b \in S$ with the following properties:

$$a \leq s \quad \forall s \in S \tag{1}$$

and

$$b \leq s \quad \forall s \in S. \tag{2}$$

Since $a \in S$, we have $b \leq a$ from (2). Similarly we obtain $a \leq b$ from (1). So $a = b$, whence uniqueness follows.

We now address the issue of existence. We shall consider two cases.

Case 1. S is a finite subset of $\mathbb{N}$.

Let us denote the cardinality of S by $|S|$. We shall apply induction on $|S|$ and show that any finite subset of $\mathbb{N}$ has a least element.

If $|S| = 1$, then this is trivial. Suppose that any set with cardinality less than or equal to n has a least element, where $n \in \mathbb{Z}_+$. Now, let S be an arbitrary subset of $\mathbb{N}$ containing $n+1$ elements. We fix an element $s_0 \in S$. If there is no element $s \in S$ satisfying $s < s_0$, then s_0 is the least element of S. If we can find an $s_1 \in S$ satisfying $s_1 < s_0$, then we look at the set $\widetilde{S} = S - \{s_0\}$. Clearly, $|\widetilde{S}| = n$. By our induction hypothesis, $\widetilde{S}$ has a least element. Since the latter element is less than or equal to s_1 and $s_1 < s_0$, we conclude that S itself contains a least element. Now, by induction, we conclude that every finite subset of $\mathbb{N}$ has a least element.

Case 2. S is an infinite subset of $\mathbb{N}$.

Since $S \neq \emptyset$, we can choose an $n \in S$ and form the finite subset $S_n = \{m \in S : m \leq n\}$ of S. Clearly, $n \in S_n \cap S$. Since S_n is finite and non-empty, Case 1 tells us that S_n contains a least element. Let s be this element. Now let us consider an element l in S. If $l \in S_n$, then clearly $s \leq l$. If $l \in S \setminus S_n$, then we have $n < l$. This relation and the fact that "$<$" is transitive imply that $s < l$. Thus s is the least element of S.

The above, together with the discussion of uniqueness, completes the proof. $\square$