MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018

QUIZ 3 **SEPTEMBER 10, 2018**

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1. Let $X =$ the set of all bounded sequences in $\mathbb{R}$ and define

$$d\big(\{x_n\},\{y_n\}\big) := \sup_{n\in\mathbb{Z}_+}|x_n - y_n|$$

on $X \times X$.

(a) (4 marks) Show that d satisfies the triangle inequality.

(b) (6 marks) Assume **without proof** that d is also positive definite and symmetric (do **not** waste time showing these; they are trivialities). Now determine whether or not the subset $S = \big\{\, \{x_n\} : x_n \in [-1,1]\ \forall n \in \mathbb{Z}_+ \big\} \subset X$ is compact.

Solution. (a) Given three sequences $\{x_n\}$ and $\{y_n\}$ and $\{z_n\}$, for each n, we have

$$|x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|. \tag{1}$$

Fix an $m \in \mathbb{Z}_+$. Now observe that

$$|x_m - z_m| \leq \sup_{k\in\mathbb{Z}_+}|x_k| + \sup_{k\in\mathbb{Z}_+}|z_k| < \infty$$

since $\{x_n\}$, $\{y_n\}$ are bounded sequences. Thus, $\sup_{k\in\mathbb{Z}_+}|x_k - z_k|$ exists and by a similar argument, $\sup_{k\in\mathbb{Z}_+}|z_k - y_k|$ exists. Thus, for any fixed n, by (1)

$$\begin{aligned}|x_n - y_n| &\leq \sup_{k\in\mathbb{Z}_+}|x_k - z_k| + \sup_{k\in\mathbb{Z}_+}|z_k - y_k| \\ &= d(\{x_l\},\{z_l\}) + d(\{z_l\},\{y_l\}).\end{aligned}$$

Thus, by the properties of the least upper bound:

$$d(\{x_l\},\{y_l\}) = \sup_{k\in\mathbb{Z}_+}|x_k - y_k| \leq d(\{x_l\},\{z_l\}) + d(\{z_l\},\{y_l\}).$$

(b) Consider the set $\{\delta^{(n)} : n \in \mathbb{Z}_+\} \subset S$, write $\delta^{(n)} = \{\delta^{(n)}_m\}_{m\in\mathbb{Z}_+}$, where

$$\delta^{(n)}_m = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise.}\end{cases} \tag{2}$$

Then by construction:

$$d(\delta^{(n)}, \delta^{(\nu)}) = 1 \text{ whenever } \nu \neq n.$$

Now consider the following open cover of S

$$\mathcal{C} := \big\{B(\{x_n\};1/2) : \{x_n\} \in S\big\}.$$

Now, suppose $\mathcal{C}$ admits a finite subcover of S, say $\mathcal{C}'$. Pick $B(\{x_n\};1/2)$ belonging to this subcover; call this open set $\mathcal{O}$. Now, if $n_{\mathcal{O}} \in \mathbb{Z}_+$ such that $\delta^{(n_{\mathcal{O}})} \in \mathcal{O}$, then consider $\nu \neq n_{\mathcal{O}}$. By (2)

$$1 = d(\delta^{(n_{\mathcal{O}})}, \delta^{(\nu)}) \leq d(\delta^{(n_{\mathcal{O}})}, \{x_n\}) + d(\{x_n\}, \delta^{(\nu)}).$$

This implies

$$\frac{1}{2} < d(\{x_n\},\, \delta^{(\nu)})).$$

As $\mathcal{O}$ denotes an arbitrary element of the above finite subcover, the last inequality shows that each $\mathcal{O}$ contains **at most** one point from the collection $\{\delta^{(n)} : n \in \mathbb{Z}_+\}$, which is infinite. This contradicts the fact that $\mathcal{C}'$ covers S. Thus $\mathcal{C}$ is an open cover admitting no finite subcover. Hence S is **not** compact.