MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018

QUIZ 4 **OCTOBER 15, 2018**

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1. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function and assume that $f(0) = f(1)$. Show that there exists a point $x_0 \in [0, 1/2]$ such that $f(x_0) = f(x_0 + \frac{1}{2})$.

Solution. Define the function $g : [0, 1/2] \to \mathbb{R}$ by

$$g(x) = f(x) - f\left(x + \tfrac{1}{2}\right).$$

We first show that $f(\cdot + \frac{1}{2})$ is a continuous function on $[0, 1/2]$. The function $[0, 1/2] \ni x \mapsto x + \frac{1}{2}$ is continuous and its range is contained in $[0,1] = \mathsf{dom}(f)$. Thus, $f(\cdot + \frac{1}{2})$, being a composition of two continuous functions, is continuous.

By the continuity of $f(\cdot + \frac{1}{2})$ and of f, we conclude that g is continuous on $[0, 1/2]$.

Now, we substitute to get

$$g(0) = f(0) - f\left(\tfrac{1}{2}\right) \quad \text{and} \quad g\left(\tfrac{1}{2}\right) = f\left(\tfrac{1}{2}\right) - f(1).$$

Since $f(0) = f(1)$, we have

$$g(0) = -g\left(\tfrac{1}{2}\right). \tag{1}$$

We now consider the following two cases.

Case (i) $g(0) = 0$.
In this case, by (1), $g(\frac{1}{2}) = 0$. Thus $x_0 = 0$ has the desired property.

Case (ii) $g(0) \neq 0$.
In this case, 0 lies between $g(0)$ and $g(\frac{1}{2})$ due to (1). As g is continuous, by the Intermediate Value Theorem, there is a point $x_0 \in (0, 1/2)$ such that $g(x_0) = 0$. This implies:

$$f(x_0) = f\left(x_0 + \tfrac{1}{2}\right).$$

This completes the proof.