MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018

QUIZ 5 **OCTOBER 29, 2018**

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1. Let $a \in \mathbb{R}$. Let $f, g : [a, +\infty) \longrightarrow \mathbb{R}$ be two continuous functions with $f(a) = g(a)$. Suppose f and g are differentiable on $(a, +\infty)$ and $f' \geq g'$ on $(a, +\infty)$. Then how are f and g related? If you feel that there is, **in general,** a relation, then give a **justification.** Alternatively, give examples in support of your answer if you feel there is no relation in general.

Solution. Consider the function $G(x) := f(x) - g(x)$, $x \in [a, +\infty)$. By hypothesis

$$G'(x) \geq 0 \ \forall x \in (a, +\infty). \tag{1}$$

Fix a point $y \in (a, +\infty)$. The function $G|_{[a,y]}$ satisfies all the properties of Lagrange's Mean Value Theorem. Thus:

$$G(y) - G(a) = G'(c)(y - a) \tag{2}$$

for some $c \in (a, y)$. By definition, $G(a) = 0$. Thus, from (1) and (2)

$$G(y) - G(a) = G(y) \geq 0.$$

Since the y above was arbitrary, we conclude that $G \geq 0$, whence $f \geq g$.