MATH 221 : ANALYSIS I – REAL ANALYSIS
AUTUMN 2018

QUIZ 6 **NOVEMBER 12, 2018**

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1. Define the function $f : \mathbb{R} \longrightarrow \{0, 1\}$ as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $f|_{[a,b]} \notin \mathcal{R}([a,b])$ for any $a < b$.

Tip. You may freely use — without proof — any fact discussed in class, or assigned for homework, about the number systems $\mathbb{Q}$ and $\mathbb{R}$.

Solution. Fix $a, b \in \mathbb{R}, a < b$. By the characterization for a bounded function to be in $\mathcal{R}([a,b])$, it suffices to show that:

$$\exists\, \epsilon_0 > 0 \,\text{such that for } \textbf{any} \text{ partition } \mathcal{P} \text{ on } [a,b],\ U(\mathcal{P}, f) - L(\mathcal{P}, f) \geq \epsilon_0. \tag{1}$$

To this end, write

$$\mathcal{P} : a = x_0 < x_1 < \cdots < x_N = b.$$

For any $j = 1, \dots, N$:
(a) by the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, x_j^* \in \mathbb{Q}$ such that $x_{j-1} < x_j^* < x_j$, whence

$$M_j := \sup_{x_{j-1} \leq x \leq x_j} f(x) = f(x_j^*) = 1.$$

(b) $[x_{j-1}, x_j]$ is in bijective correspondence with $[0, 1]$, whence it is uncountable. As $\mathbb{Q} \cap [x_{j-1}, x_j]$ is countable, $\exists\, x_j' \in [x_{j-1}, x_j] - \mathbb{Q}$. Thus

$$m_j := \inf_{x_{j-1} \leq x \leq x_j} f(x) = f(x_j') = 0.$$

From (a) and (b), we have

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{j=1}^{N} (M_j - m_j) \triangle x_j = b - a.$$

Clearly, (1) holds true for any $\epsilon_0 \in (0, b - a)$. Thus $f \notin \mathcal{R}([a,b])$.