MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2020

HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: MAY 15, 2020**

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1. Let $(X, \mathcal{F})$ be a measurable space and let μ_1, μ_2 be two measures on $\mathcal{F}$. Suppost **at least one** of $\{\mu_1, \mu_2\}$ is a finite measure. Show that the set function defined by

$$\nu(E) := \mu_1(E) - \mu_2(E) \ \ \forall E \in \mathcal{F}$$

is a signed measure.

2. Let $(X, \mathcal{F})$ be a measurable space and let ν be a complex measure on it. Show that $\mathsf{Re}(\nu)$ and $\mathsf{Im}(\nu)$ are signed measures.

3. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded monotone function. Show that F is of bounded variation and give a formula for its total variation $\Lambda(F)$.

4. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a periodic function — i.e., $\exists T \in (0, +\infty)$ such that $F(x+T) = F(x)$ $\forall x \in \mathbb{R}$. When is F of bounded variation?
Note. This problem shows that being of bounded variation is quite a restrictive condition. While the class BV contains many functions that are not even continuous (e.g., see the previous problem), plenty of **very well-behaved** functions fail to be of bounded variation.

5. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$.

(a) Let $a < b \in \mathbb{R}$. Show that

$$V_F(b) - V_F(a) \ = \ \sup\left\{\sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| : N \in \mathbb{Z}_+, \ a = x_0 < x_1 < \cdots < x_N = b\right\}.$$

(b) Since the right-hand side of the above identitiy depends **only** on the values F takes on $[a, b]$:

- We call $\big(V_F(b) - V_F(b)\big)$ the *total variation of* F *on* $[a, b]$, which we denote by $\Lambda(F; [a, b])$.
- We say that a function $f : [a, b] \longrightarrow \mathbb{R}$ is of bounded variation, denoted $f \in BV[a, b]$, if and only if $\Lambda(f; [a, b]) < \infty$.

Now define

$$F(x) := \begin{cases} x\sin(1/x), & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that $F|_{[a,\,b]} \notin BV[a, b]$ if $a = 0$.

6. Making use of the Hahn Decomposition Theorem, prove the following version of the Jordan Decomposition Theorem:
Let $(X, \mathcal{F})$ be a measurable space and let ν be a signed measure on it. Show that there exists a unique pair of measures (μ_ν^+, μ_ν^-) such that

$$\nu = \mu_\nu^+ - \mu_\nu^- \quad \textit{and} \quad \mu_\nu^+ \perp \mu_\nu^-.$$