MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2020
HOMEWORK 11

Instructor: GAUTAM BHARALI **Assigned: MAY 22, 2020**

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1. Let $(X, \mathcal{F})$ be a measurable space and let ν be a signed measure on it. Let $\mu_\nu^\pm$ be the (positive) measures associated to ν given by its Jordan decomposition. Consider the positive measure

$$|\nu| := \mu_\nu^+ + \mu_\nu^-.$$

Let $f : \mathcal{F} \longrightarrow \mathbb{R}$ be a measurable function. Show that $f \in \mathbb{L}^1(X, \nu)$ if and only if $f \in \mathbb{L}^1(X, |\nu|)$.
Remark. The measure $|\nu|$ is called the *total variation of* ν.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space. **Assuming** that the statement of the Lebesgue Decomposition Theorem is true under the additional assumption that ν — as in the the statement of that theorem — is a (positive) measure, prove the general statement of the Lebesgue Decomposition Theorem.

3. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\nu_1, \nu_2 : \mathcal{F} \longrightarrow [0, +\infty)$ be two finite measures such that

$$\nu_1 \perp \mu \quad \text{and} \quad \nu_2 \perp \mu.$$

Show that $(\nu_1 - \nu_2) \perp \mu$.

4. Let $(X, \mathcal{F})$ be a measurable space and let μ and ν be σ-finite measures on it with $\nu \ll \mu$. Let $\lambda = \mu + \nu$. If $f := d\nu/d\lambda$, then show that $0 \leq f \leq 1$ μ-a.e. and $d\nu/d\mu = f/(1-f)$.

5. Let $(X, \mathcal{F})$ be a measurable space and let ρ and τ be **finite** measures on it. Show that either $\rho \perp \tau$ or there exists a $\delta > 0$ and a set $A \in \mathcal{F}$ such that $\tau(A) > 0$ and

$$\rho(E) \geq \delta\tau(E) \;\; \forall E \subseteq A : E \in \mathcal{F}.$$

6. Let $(X, \mathcal{F})$ be a measurable space and let μ and ν be measures on it. Show that $\nu \ll \mu$ if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $E \in \mathcal{F}$ satisfying $\mu(E) < \delta$, we have $\nu(E) < \varepsilon$.