MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2020

HOMEWORK 12

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Note: In what follows, if $(X, \mathcal{F}, \mu)$ is a measure space, then $\mathbb{L}^p(X, \mu)$ — without mention of the underlying field — will denote the $\mathbb{L}^p$-space arising from $\mathbb{R}$-valued measurable functions.

1. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let ν be a σ-finite signed measure on $\mathcal{F}$. Show that for any $E \in \mathcal{F}$ such that $\chi_E \in \mathbb{L}^1(X, \nu)$,

$$\chi_E \left(\frac{d\nu}{d\mu}\right) \in \mathbb{L}^1(X, \mu), \text{ and}$$

$$\int_X \chi_E \, d\nu = \int_X \chi_E \left(\frac{d\nu}{d\mu}\right) d\mu.$$

2. Let $(V, \|\cdot\|)$ be a complete normed linear space (with respect to the metric induced by $\|\cdot\|$). Show that $(V^*, \|\cdot\|_{V^*})$ is complete.

Remark. A normed linear space that is complete with respect to the distance induced by the norm is called a *Banach space*.

3. Let $(X, \mathcal{F})$ be a measurable space and let

$$M(\mathcal{F}) := \{\nu : \mathcal{F} \longrightarrow \mathbb{R} \mid \nu \text{ is a signed measure}\}.$$

In view of the fact that each $\nu \in M(\mathcal{F})$ takes values in $\mathbb{R}$, we can define

$$\nu + \rho \quad \text{and} \quad c\nu \;\; \forall \nu, \rho \in M(\mathcal{F}) \text{ and } \forall c \in \mathbb{R}$$

in the obvious manner. Define $\|\nu\| := |\nu|(X)$. Show that $(M(\mathcal{F}), \|\cdot\|)$ is a normed linear space over $\mathbb{R}$.

Remark. The norm $\|\cdot\|$ on $M(\mathcal{F})$ defined above is called the *total-variation norm*.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space and fix $p : 1 \leq p \leq +\infty$. Let q denote the conjugate exponent. Fix a $g \in \mathbb{L}^q(X, \mu)$, write

$$\lambda([f]) := \int_X fg \, d\mu \;\; \forall f \text{ belonging to } \mathbb{L}^p(X, \mu),$$

and show that (i) the integral on the right-hand side exists; (ii) the left-hand side is well-defined (i.e., is independent of the choice of the representative of $[f] \in \mathbb{L}^p(X, \mu)$) and makes $\lambda : \mathbb{L}^p(X, \mu) \longrightarrow \mathbb{R}$ a bounded linear functional; and (iii) $\|\lambda\|_{(\mathbb{L}^p)^*} \leq \|g\|_q$.

5. Define $\mathcal{C}_0(\mathbb{R}^n; \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, to be the closure of $\mathcal{C}_c(\mathbb{R}^n; \mathbb{F})$ with respect to the metric induced by the $\mathbb{L}^\infty$-norm. Show that

$$\mathcal{C}_0(\mathbb{R}^n; \mathbb{F}) = \left\{ f : \mathbb{R}^n \longrightarrow \mathbb{F} \mid f \text{ is continuous, and } \lim_{\|x\| \to +\infty} f(x) = 0 \right\}$$

6. Let $(X_i, \mathcal{F}_i)$ be measure spaces and let μ_i, ν_i be σ-finite (positive) measures on $\mathcal{F}_i$, $i = 1, 2$. Suppose $\nu_i \ll \mu_i$, $i = 1, 2$. Show that $(\nu_1 \times \nu_2) \ll (\mu_1 \times \mu_2)$. Now, deduce a formula for

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}$$

in terms of the Radon–Nikodym derivatives of ν_i with respect to μ_i, $i = 1, 2$.