MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2020

HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: JANUARY 17, 2020**

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1. State whether the following is **true or false** (and justify): If S is a non-empty finite set, then there are no σ-algebras on S other than $\{\varnothing, S\}$ and $\mathscr{P}(S)$.

2. Let X be an uncountable set and define

$$\mathcal{F} := \{A \in \mathscr{P}(X) : \text{either } A \text{ or } X \setminus A \text{ is at most countable}\}.$$

Show that $\mathcal{F}$ is a σ-algebra.

3. Let $(X, \mathcal{F})$ be a measurable space and let $\mu : \mathcal{F} \longrightarrow [0, +\infty]$. Consider the following properties:

(M0) $\mu(\varnothing) = 0$;

(M1) For any $A_1, \dots, A_N \in \mathcal{F}$ that are pairwise disjoint $\mu\big(\cup_{1 \leq j \leq N} A_j\big) = \sum_{1 \leq j \leq N} \mu(A_j)$;

(M2) For any increasing sequence of sets $\{A_n\} \subset \mathcal{F}$, $\mu\big(\cup_{n=1}^{\infty} A_n\big) = \lim_{n \to \infty} \mu(A_n)$;

and a related property:

(M1$'$) For any sequence of pairwise disjoint sets $\{A_n\} \subset \mathcal{F}$, $\mu\big(\cup_{n=1}^{\infty} A_n\big) = \sum_{n=1}^{\infty} \mu(A_n)$.

Show that μ has the properties (M0), (M1) and (M2) if and only if it has the properties M(0) and (M1$'$).

4. (Restriction of a measure) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $E \in \mathcal{F}$. Define $\mathcal{F}|_E := \{E \cap A : A \in \mathcal{F}\}$ and let $\mu|_E$ denote the restriction of μ to $\mathcal{F}|_E$. Show that $(E, \mathcal{F}|_E, \mu|_E)$ is a measure space.

5. Let $X \neq \varnothing$ and let $\mathcal{F} = \mathscr{P}(X)$. Let $x_1, \dots, x_N$ be distinct points in X. Let $a_1, \dots, a_N > 0$. Define

$$\mu(A) := \sum_{j=1}^{N} \lambda_j(A) \quad \forall A \in \mathcal{F},$$

where

$$\lambda_j(A) := \begin{cases} 1, & \text{if } x_j \in A, \\ 0. & \text{otherwise.} \end{cases}$$

Show that μ is a measure.

Remark. When X is a manifold, $\mathcal{F}$ is the Borel σ-algebra on X, $N = 1$ and $a_1 = 1$, then the above μ is called the **Dirac mass at $\boldsymbol{x_1}$.**

6. Let $\{A_j\}$ be an increasing sequence of subsets of $\mathbb{R}^n$. Show that $m^*\big(\cup_{j=1}^{\infty} A_j\big) \leq \lim_{j \to \infty} m^*(A_j)$.

7. Fix a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ that is increasing (not necessarily strictly) and continuous from the **right.** For any $A \subseteq \mathbb{R}$, define

$$\mu_F^*(A) := \inf \Big\{ \sum_j (F(b_j) - F(a_j)) : \{(a_j, b_j] : j \in J\} \text{ is an admissible} \\ \text{cover of } A,\ a_j, b_j \in \mathbb{R} \text{ and } a_j < b_j \Big\}.$$

Show that μ_F^* is an outer measure.