MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2020

HOMEWORK 2

Instructor: GAUTAM BHARALI **Assigned: JANUARY 24, 2020**

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1. Prove the following:

Theorem (Carathéodory)**.** *Let X be an infinite set and let μ^* be an outer measure on X. Define*

$$M(\mu^*) := \{E \subset X : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) \text{ for every } A \in \mathscr{P}(X)\},$$

and set $\mu := \mu^*|_{M(\mu^*)}$. Then $(X, M(\mu^*), \mu)$ is a measure space.

2. Fix a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ that is increasing (not necessarily strictly) and continuous from the **right.** Recall, from Problem 7 of Homework 1, the **outer measure** on $\mathbb{R}$ given by

$$\mu_F^*(A) := \inf\Big\{ \sum_j (F(b_j) - F(a_j)) : \{(a_j, b_j] : j \in J\} \text{ is an admissible cover of } A,\ a_j, b_j \in \mathbb{R} \text{ and } a_j < b_j \Big\},$$

where A is any subset of $\mathbb{R}$. Let $M(\mu_F^*)$ be the σ-algebra associated to μ_F^* given by the Carathéodory construction above. We can show that $\mu_F^*((a, b]) = F(b) - F(a)$; **assuming this** if necessary, show that $\mathscr{B}(\mathbb{R}) \subset M(\mu_F^*)$.

3. (Completion of a measure) Let $(X, \mathcal{F}, \mu)$ be a measure space. Define the family

$$\mathcal{N}(\mu) := \{Z \subset X : \exists A \in \mathcal{F} \text{ such that } A \supset Z \text{ and } \mu(A) = 0\}.$$

(a) Show that

$$\overline{\mathcal{F}} := \{A \cup Z : A \in \mathcal{F},\ Z \in \mathcal{N}(\mu)\}$$

is a σ-algebra.

(b) Show that μ has a **unique** extension to a measure $\overline{\mu} : \overline{\mathcal{F}} \longrightarrow [0, +\infty]$.

(c) Show that $(X, \overline{\mathcal{F}}, \overline{\mu})$ is complete.

Remark. The measure space $(X, \overline{\mathcal{F}}, \overline{\mu})$ is called the *completion of* $(X, \mathcal{F}, \mu)$.

4. (Large Cantor sets) Consider a sequence $\{\alpha_1, \alpha_2, \alpha_3, \dots\} \subset (0, 1)$. Construct a Cantor-like set as follows: Let $K_0 := [0, 1]$. For each $n = 1, 2, 3, \dots$, define:

$K_n :=$ the set obtained by removing open intervals that form the middle α_n^{th} fraction of each connected component of K_{n-1}.

Define

$$K := \bigcap_{n=1}^{\infty} K_n.$$

(a) Show that K has empty interior (**Note:** We know by now how to argue that K is non-empty!)

(b) Show that K is Lebesgue measurable.

(c) Show that $m(K) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_n < +\infty$.

Hint. You will need a standard observation about infinite products. The tricky part in solving (c) is an argument in classical analysis linking the latter observation with the series $\sum_{j=1}^{\infty} \alpha_n$.

5. Show that, given any Lebesgue-measurable set $E \subsetneq \mathbb{R}$ with $m(E) > 0$, there exists a **non-measurable** set $A \subset E$.

6. Fix $n \in \mathbb{Z}_+$. Let $E \subset \mathbb{R}^n$. Using the fact that

($*$) $E \in \mathscr{M}_n \Rightarrow$ given any $\varepsilon > 0$, there exists an open set $\Omega_\varepsilon \subset \mathbb{R}^n$ such that $\Omega_\varepsilon \supsetneq E$ and $m^*(\Omega_\varepsilon \setminus E) < \varepsilon$

for E such that $m^*(E) < \infty$, show that ($*$) holds true for the case $m^*(E) = \infty$.

7. Recall that:

- Given a topological space X, a collection of open sets $\mathscr{C}$ is a *basis* for the prescribed topology if for each $x \in X$ and each open set $U \ni x$, there exists an open set $V \in \mathscr{C}$ such that $x \in V \subset U$.
- An *open cube* in $\mathbb{R}^n$ is a set of the form

$$\prod_{i=1}^{n} (a_i - \varepsilon, a_i + \varepsilon),$$

where $(a_1, \dots, a_n) \in \mathbb{R}^n$ and $\varepsilon > 0$.

Fix an $n \in \mathbb{Z}_+$, $n \geq 2$.

(a) Show that $\mathbb{R}^n$ has a countable basis comprising open cubes.

(b) Assuming that, for any $(a_1, \dots, a_n) \in \mathbb{R}^n$, the orthants

$$\prod_{i=1}^{n} (-\infty, a_i) \quad \text{and} \quad \prod_{i=1}^{n} (a_i, +\infty)$$

belong to $\mathscr{M}_n$, show that the Borel σ-algebra $\mathscr{B}(\mathbb{R}^n) \subset \mathscr{M}_n$.