# MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

**SPRING 2020**

**HOMEWORK 4**

**Instructor: GAUTAM BHARALI** **Assigned: FEBRUARY 7, 2020**

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**1.** Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $E \in \mathcal{F}$ is such that $\mu(E) = 0$. Let $f : X \longrightarrow [-\infty, +\infty]$ be such that $f|_{X \setminus E}$ is measurable. Show that there exists a $g : X \longrightarrow [-\infty, +\infty]$ that is measurable such that $f|_{X \setminus E} = g|_{X \setminus E}$.

**2.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $A \in \mathcal{F}$. Let $\phi : X \longrightarrow \mathbb{R}$ be a simple non-negative measurable function. Show that

$$\int_A \phi \, d\mu = \int_X \phi \chi_A \, d\mu.$$

**3.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $A \in \mathcal{F}$. Let $f : A \longrightarrow [-\infty, +\infty]$ be a measurable function. Does there exist a measurable function $g : X \longrightarrow [-\infty, +\infty]$ such that $g|_A = f$?

**4.** Equip $\mathbb{R}$ with the Borel $\sigma$-algebra. Let $x_1, \dots, x_N$ be distinct points in $\mathbb{R}$. Let $a_1, \dots, a_N > 0$. For any $x \in \mathbb{R}$, define the measure $\delta_x$, known as the **Dirac mass at $x$**, by

$$\delta_x(E) := \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{otherwise,} \end{cases}$$

for every $E \in \mathscr{B}(\mathbb{R})$. (We have seen in Homework 1 that $\delta_x$ is a measure in this setting.) Define

$$\mu := \sum_{j=1}^{N} a_j \delta_{x_j}.$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel-measurable function. Follow the step-by-step construction of the Lebesgue integral to derive a formula for $\int_{\mathbb{R}} f \, d\mu$.

**5.** Show that the Monotone Convergence Theorem can be deduced from Fatou's Lemma.

**6.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \longrightarrow [0, +\infty]$ be measurable. Show that

$$\int_X f \, d\mu = 0 \iff f = 0 \text{ a.e.}$$

**Hint.** First consider the case when $f$ is a simple non-negative measurable function.

**7.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \longrightarrow [-\infty, +\infty]$ be measurable. show that $f$ is Lebesgue integrable if and only if

$$\int_X |f| \, d\mu < +\infty.$$

**8.** (Chebyshev's inequality) Let $(X, \mathcal{F}, \mu)$ be a measure space and $A \in \mathcal{F}$. Suppose $f : A \longrightarrow [0, +\infty]$ and that $f$ is integrable. Let $\alpha > 0$. Show that

$$\mu\left[\{f > \alpha\}\right] \le \alpha^{-1} \int_E f \, d\mu.$$