MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2020

HOMEWORK 5

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 14, 2020**

This assignment features a **proper subset** of the subjects studied in the lectures of the week of February 10–14. It also features miscellaneous problems representative of the syllabus of the mid-semester examination.

1. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $A \in \mathcal{F}$. Let $f : X \longrightarrow [-\infty, +\infty]$ be a measurable function and suppose f is Lebesgue integrable on A. Show that the Lebesgue integral $\int_A f\, d\mu$ as defined in class agrees with the the integral of $f|_A$ that is obtained by:

- starting with the measure space $(A, \mathcal{F}|_A, \mu|_A)$ and viewing $f|_A$ as a measurable function relative to $(A, \mathcal{F}|_A)$; and
- retracing the entire step-by-step process of constructing the integral on the measure space $(A, \mathcal{F}|_A, \mu|_A)$.

2. Let $E \in \mathscr{M}_1$ such that $m(E) > 0$. Show that for each $c \in (0, 1)$, there exists an open interval I_c such that $m(I_c \cap E) \geq cm(I_c)$.

3. Construct a Borel set E in $\mathbb{R}$ such that for *any* non-empty open interval $I \subset \mathbb{R}$, $0 < m(E \cap I) < m(I)$.
Hint. Start with a "large Cantor set" as described in Homework 2 and use an inductive procedure to obtain an E with the desired properties.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of non-negative measurable functions on X. Suppose there exists a function $f : X \longrightarrow [0, +\infty]$ such that $f_n \longrightarrow f$ a.e. and $f \geq f_n$ a.e. for each $n \in \mathbb{Z}_+$. Show that

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu.$$

5. Make use of the Cantor–Lebesgue function in an appropriate way to show that it is **not** true that for any $E \in \mathscr{M}_1$, and any $f : E \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ that are (Lebesgue) measurable, $g \circ f$ is measurable.