# MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
## SPRING 2020
## HOMEWORK 6

**Instructor: GAUTAM BHARALI** **Assigned: FEBRUARY 28, 2020**

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**1.** Let $\{(X_\alpha, \mathcal{F}_\alpha)\}_{\alpha\in A}$ be an indexed family of measurable spaces. Assume that $A$ is at most countable. Show that:

(*i*) $\otimes_{\alpha\in A}\mathcal{F}_\alpha$ is generated by the collection

$$\Big\{\prod\nolimits_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{F}_\alpha \ \forall \alpha \in A\Big\}.$$

(*ii*) If $\mathcal{F}_\alpha$ is generated by $\mathcal{C}_\alpha \subset \mathscr{P}(X_\alpha)$, then $\otimes_{\alpha\in A}\mathcal{F}_\alpha$ is generated by the collection

$$\Big\{\prod\nolimits_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{C}_\alpha \ \forall \alpha \in A\Big\}.$$

In the two problems that follow, we shall use the following notation. Firstly: $m_N^*$, $\mathscr{M}_N$ and $m_N$ will denote, respectively, the Lebesgue outer measure, the Lebesgue $\sigma$-algebra, and the Lebesgue measure — as defined in class — on $\mathbb{R}^N$, $N \in \mathbb{Z}_+$.

Next: consider the product measure on $\mathbb{R}^N$ arising from $(\mathbb{R}, \mathscr{M}_1, m_1)$ and let $\pi$ and $\pi^*$ be the set functions (the latter an outer measure) associated to the product construction. Define, for $A \subset \mathbb{R}^N$:

$$\begin{aligned}
\mathscr{C}_Q(A) &:= \text{the collection of covers of } A \text{ admissible in the definition of } m_N^*,\\
\mathscr{C}_{box}(A) &:= \text{the collection of covers of } A \text{ admissible in the definition of } \pi^*,\\
M(\pi^*) &:= \text{the } \sigma\text{-algebra arising from applying the Carathéory condition to } \pi^*.
\end{aligned}$$

**2.** Let $A \subset \mathbb{R}^N$ and suppose $\pi^*(A) < \infty$. Given an $\varepsilon > 0$, let $\{B_j : j \in J\} \in \mathscr{C}_{box}(A)$ be a cover of $A$ such that $\sum_{j\in J}\pi(B_j) < \pi^*(A) + \varepsilon$. Show that there is a cover $\{Q_n : n \in \mathbb{Z}_+\} \in \mathscr{C}_Q(A)$ such that

$$\sum_{n=1}^{\infty}\mathrm{vol}(Q_n) \ < \ \pi^*(A) + C\varepsilon$$

for some constant $C > 0$ that does **not** depend on $\{B_j : j \in J\}, \{Q_n : n \in \mathbb{Z}_+\}$.
**Hint.** First reduce the problem to a basic, simply-stated claim about the **geometry** of open sets in $\mathbb{R}^N$, and prove the above using this claim. Thereafter, try to prove the claim itself.

**3.** How are $\mathscr{M}_N$ and $M(\pi^*)$ related?

**4.** Let $X$ be a non-empty set and suppose $\mathcal{A} \subseteq \mathscr{P}(X)$ is an algebra. Let $\mathscr{C}(\mathcal{A})$ denote the monotone class generated by $\mathcal{A}$. Show that $\mathscr{C}(\mathcal{A}) = \mathcal{F}(\mathcal{A})$.

**5.** Let $E \in \mathscr{M}_1$ and let $f : E \longrightarrow [0, +\infty)$ be a non-negative Lebesgue-measurable function. Show that the set

$$S := \{(x, y) \in E \times \mathbb{R} : 0 \le y \le f(x) \ \forall x \in E\}$$

belongs to $\mathscr{M}_1 \otimes \mathscr{M}_1$.

**6.** Let $-\infty < a < b < +\infty$, write $I := [a, b]$, an interval in $\mathbb{R}$, and let $\phi : I \longrightarrow \mathbb{R}$ be a continuous, strictly increasing function. Define:

$$S := \{(x, y) \in E \times \mathbb{R} : 0 \leq y \leq \phi(x) \; \forall x \in E\}.$$

Let $f : S \longrightarrow \mathbb{R}$ be in $\mathbb{L}^1(S, (m \times m)|_S)$. State and prove the intermediate assertions needed to make sense of the following statement:

$$\begin{aligned}\int_S f \, d(m \times m) &= \int_I \left[ \int_{[0, \phi(x)]} f(x, y) \, dm(y) \right] dm(x) \\ &= \int_{\phi(I)} \left[ \int_{[\phi^{-1}(y), b]} f(x, y) \, dm(x) \right] dm(y).\end{aligned}$$

Then, prove the above statement.

**Clarification:** Do **not** attempt a solution beginning with an auxiliary statement involving simple functions! Using the conclusions of problems stated in previous assignments, **if** necessary, try to reduce the problem to a suitable application of the Tonelli / Fubini Theorem.