MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2020
HOMEWORK 7

Instructor: GAUTAM BHARALI **Assigned: MARCH 7, 2020**

---

1. Let $\{(X_\alpha, \mathcal{F}_\alpha)\}_{\alpha\in A}$ be an indexed family of measurable spaces. Assume that A is at most countable. The statement asked to be proved in **Problem 1-(*ii*)** in the last assignment is **false.** Establish the following **correction** of the latter statement:

> If, for each $\alpha \in A$, $\mathcal{F}_\alpha$ is generated by $\mathcal{C}_\alpha \subset \mathscr{P}(X_\alpha)$ with the property that $X_\alpha \in \mathcal{C}_\alpha$, then $\otimes_{\alpha\in A}\mathcal{F}_\alpha$ is generated by the collection
> $$\Big\{ \prod\nolimits_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{C}_\alpha \ \forall \alpha \in A \Big\}.$$

Note: In what follows, if $(X, \mathcal{F}, \mu)$ is a measure space, then $\mathbb{L}^p(X, \nu)$ — without mention of the underlying field — will denote the $\mathbb{L}^p$-space arising from $\mathbb{R}$-valued measurable functions.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space. A set $E \in \mathcal{F}$ is called an **atom** if $\mu(E) > 0$ and contains no measurable subset of strictly smaller but positive measure. We say that $(X, \mathcal{F}, \mu)$ is **non-atomic** if there are no atoms in $\mathcal{F}$.

Let $(X, \mathcal{F}, \mu)$ be a non-atomic and σ-finite measure space (where, to avoid trivialities, we take $\mu(X) > 0$). Show that the conclusion of Minkowski's inequality is false on $\mathbb{L}^p(X, \mu)$ for each $p \in (0, 1)$.

3. Show that $\mathbb{L}^\infty(\mathbb{R}^n, m)$ is not separable.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $0 < p < r \leq \infty$. Show, using Hölder's inequality **appropriately** (note that it is possible that $0 < p < 1$), that $\mathbb{L}^r(X, \mu, \mathbb{F}) \subset \mathbb{L}^p(X, \mu, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

5. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $0 < p < r \leq \infty$. Show, **without** using Hölder's inequality, that $\mathbb{L}^r(X, \mu, \mathbb{F}) \subset \mathbb{L}^p(X, \mu, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

6. (**Minkowski's Integral Inequality**) Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be σ-finite measure spaces and let f be a $\mathscr{M} \otimes \mathscr{N}$-measurable $[0, +\infty)$-valued function on $X \times Y$. Let $1 \leq p < \infty$. Show that

$$\left[\int_X \left|\int_Y f(x,y)\, d\nu(y)\right|^p d\mu(x)\right]^{1/p} \ \leq \ \int_Y \left[\int_X f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y).$$