MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2020

HOMEWORK 8

Instructor: GAUTAM BHARALI **Assigned: MARCH 13, 2020**

Note: In what follows, if $(X, \mathcal{F}, \mu)$ is a measure space, then $\mathbb{L}^p(X, \mu)$ — without mention of the underlying field — will denote the $\mathbb{L}^p$-space arising from $\mathbb{R}$-valued measurable functions.

1. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $1 \leq p < \infty$. Let f_n, f, g_n, g be **real**-valued measurable functions such that

- $g_n \longrightarrow g$ a.e. and $\|g_n\|_\infty \leq M < \infty \ \forall n \in \mathbb{Z}_+$.
- $\{f_n\}$ is a sequence in $\mathbb{L}^p(X, \mu)$ and $f_n \longrightarrow f$ in $\mathbb{L}^p$-norm.

Show that $f_n g_n \longrightarrow fg$ in $\mathbb{L}^p$-norm.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let f_n, g_n, g, f be **real**-valued measurable functions, $n = 1, 2, 3, \dots$ Suppose $f_n \longrightarrow f$ in measure and $g_n \longrightarrow g$ in measure. Does it follow that $f_n g_n \longrightarrow fg$ in measure? If your answer is, "No," then:

- Provide a **convincing counterexample**; and
- Impose some reasonable hypothesis under which $f_n g_n \longrightarrow fg$ in measure.

3. Let $(X, \mathcal{F}, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of measurable $\mathbb{F}$-valued functions, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let f be a measurable $\mathbb{F}$-valued function. Show that if $f_n \longrightarrow f$ almost uniformly, then $f_n \xrightarrow{\mu} f$.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence in $\mathbb{L}^1(X, \mu)$ such that $|f_n| \leq g$ for some g in $\mathbb{L}^1(X, \mu)$. Suppose $f : X \longrightarrow \mathbb{R}$ is a measurable function such that $f_n \xrightarrow{\mu} f$. Then, show that f is Lebesgue integrable, and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

5. Give an **explicit** expression of a compactly-supported function that is of class $\mathcal{C}^\infty(\mathbb{R}^n)$ and whose support is the Euclidean unit ball in $\mathbb{R}^n$.

Hint. First tackle the above problem for the real line.

6. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{E_n\} \subset \mathcal{F}$ such that $\mu(E_n) < \infty \ \forall n \in \mathbb{Z}_+$. Let $f \in \mathbb{L}^1(X, \mu)$ and suppose $\chi_{E_n} \longrightarrow f$ in $\mathbb{L}^1$-norm. Show that f is a.e. equal to the characteristic functin of some measurable set.