MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2020

HOMEWORK 9

Instructor: GAUTAM BHARALI **Assigned: MAY 8, 2020**

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1. This problem was **originally assigned in Homework 8.** But since the mode ofconvergence termed *almost uniform convergence* had not then been defined, it is being repeated:

Let $(X, \mathcal{F}, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of measurable $\mathbb{F}$-valued functions, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let f be a measurable $\mathbb{F}$-valued function. Show that if $f_n \longrightarrow f$ almost uniformly, then $f_n \xrightarrow{\mu} f$.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of $\mathbb{F}$-valued measurable functions, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let f be a measurable $\mathbb{F}$-valued function. Show that if $f_n \xrightarrow{\mu} f$, then $\{f_n\}$ is Cauchy in measure.

3. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two (Lebesgue) measurable functions. Recall the meaning of the expression $f * g$ *exists* introduced in class. Show that

$$f * g \text{ exists} \quad \Longleftrightarrow \quad |f| * |g|(x) < +\infty \text{ for a.e. } x \in \mathbb{R}^n.$$

4. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be (Lebesgue) measurable. Show that the function

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x, t) \longmapsto f(x - t)$$

is $\mathscr{M}_n \times \mathscr{M}$-measurable.

5. Let $f \in \mathbb{L}^\infty(\mathbb{R}^n)$ and let $g \in \mathbb{L}^p(\mathbb{R}^n)$, where $1 < p < \infty$. If f is compactly supported, then show that $f * g(x)$ exists and is finite for every $x \in \mathbb{R}^n$.

6. Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ belong to $\mathbb{L}^1(\mathbb{R}^n$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. Fix a positive constant δ. Then, show that

$$\lim_{\varepsilon \to 0^+} \int_{\{x : \|x\| > \delta\}} |K_\varepsilon| \, dx \, = \, 0.$$

7. Show that if $f \in \mathbb{L}^p(\mathbb{R}^n)$ and $g \in \mathbb{L}^q(\mathbb{R}^n)$, where $1 \leq p \leq +\infty$ and p and q are conjugate exponents, then $f * g$ is a **continuous** function on $\mathbb{R}^n$.