MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2020
"THE CORONAVIRUS VACATION" REVIEW ASSIGNMENT # 1

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1. Show that there exists a Borel measurable set $A \subset \mathbb{R}$ such that $0 < m(A \cap I) < m(I)$ for **every** non-empty interval I.

Remark. Problem 4 of Assignment 2 (i.e., on "large Cantor sets") might be helpful.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{A_n\}$ be a sequence in $\mathcal{F}$. Suppose the series

$$\sum_{n=1}^{\infty} \mu(A_n)$$

converges. Then, show that for μ-a.e. $x \in X$, x belongs to at most finitely many of the E_j's.

3. Fix an $n \in \mathbb{Z}_+$. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an invertible linear transformation. You may freely use **without proof** the fact that for any set $Q \subseteq \mathbb{R}^n$ of the form $Q = I_1 \times \cdots \times I_n$, where each I_i, $1 \leq i \leq n$, is a non-empty interval,

$$\mathsf{vol}\big(T(Q)\big) = m^*\big(T(Q)\big) = m\big(T(Q)\big) = |\det(T)|\mathsf{vol}(Q).$$

Using this fact:

(a) Show that for any $E \in \mathscr{M}_n$, $T(E) \in \mathscr{M}_n$ and $m(T(E)) = |\det(T)|m(E)$.

(b) Show that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is (Lebesgue) measurable, then so is $f \circ T$.

(c) Use (a) and (b) to show that if ϕ is a simple non-negative measurable function, then

$$\int_{\mathbb{R}^n} (\phi \circ T)\, dm = |\det(T)|^{-1} \int_{\mathbb{R}^n} \phi\, dm.$$

(d) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lebesgue integrable. Then, show that

$$\int_{\mathbb{R}^n} (f \circ T)\, dm = |\det(T)|^{-1} \int_{\mathbb{R}^n} f\, dm.$$

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue measurable 1-periodic function (i.e., $f(x+1) = f(x)$ for every $x \in \mathbb{R}$). Suppose $\exists C \in (0, \infty)$ such that

$$\int_{[0,1]} |f(a+x) - f(b+x)|\, dm(x) \leq C \quad \forall a, b \in \mathbb{R}.$$

Show that $f|_{(0,1)}$ is Lebesgue integrable on $(0, 1)$.

Note: In what follows, if $(X, \mathcal{F}, \mu)$ is a measure space, then $\mathbb{L}^p(X, \mu)$ — without mention of the underlying field — will denote the $\mathbb{L}^p$-space arising from $\mathbb{R}$-valued measurable functions.

5. Consider the measure space $([-1,1], \mathscr{M}_1|_{[-1,1]}, m|_{[-1,1]})$. Let $\{f_n\}$ be a sequence in $\mathbb{L}^2([-1,1], m)$ and let $f_n : [-1,1] \longrightarrow [0, \infty)$ for each $n \in \mathbb{Z}_+$. Assume that $\|f_n\|_1 = 2$ and

$$\big|\, \|f_n\|_2 - \sqrt{2}\,\big| \;\leq\; 2^{-n}$$

for each $n \in \mathbb{Z}_+$. Show that $\{f_n\}$ converges a.e.; what is this limit function?

6. For $n \in \mathbb{Z}_+$, let V_n denote the volume of the closed unit ball in $\mathbb{R}^n$ with centre 0. Show, by the use of either the Fubini or the Tonelli theorem, that

$$V_n \;=\; 2V_{n-1} \int_{[0,1]} (1 - x^2)^{(n-1)/2} dm(x) \quad \text{for } n = 2, 3, 4, \ldots$$

Please be sure to provide **complete** justifications!

7. Consider the measure space $([0, 2\pi], \mathscr{M}_1|_{[0,2\pi]}, m|_{[0,2\pi]})$ and fix a function f in $\mathbb{L}^2([0, 2\pi], m)$. Does either of the sequences

$$\left\{ \int_{[0,2\pi]} f(x) \cos(nx)\, dm(x) \right\},$$

$$\left\{ \int_{[0,2\pi]} f(x) \sin(nx)\, dm(x) \right\}$$

converge? If so, then determine the limit — giving complete justifications.