MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2020
"THE CORONAVIRUS VACATION" REVIEW ASSIGNMENT # 2

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Note: In what follows, if $(X, \mathcal{F}, \mu)$ is a measure space, then $\mathbb{L}^p(X, \mu)$ — without mention of the underlying field — will denote the $\mathbb{L}^p$-space arising from $\mathbb{R}$-valued measurable functions.

1. Let $a < b \in [-\infty, +\infty]$ and let $g : (a, b) \longrightarrow \mathbb{R}$ be a *convex* function: which means that

$$g((1-t)x + ty) \leq (1-t)g(x) + tg(y) \quad \forall x, y \in (a, b) \text{ and } \forall t \in [0, 1].$$

(a) Show that g is (Lebesgue) measurable.

(b) Let $(X, \mathcal{F}, \mu)$ be a measure space, and suppose $\mu(X) = 1$. Let $f : X \longrightarrow (a, b)$ belong to $\mathbb{L}^1(X, \mu)$. Show that $g \circ f$ is Lebesgue integrable in the extended sense and that

$$g\left(\int_X f\, d\mu\right) \leq \int_X (g \circ f)\, d\mu.$$

Hint. What property of g, if demonstrated, would **simultaneously** establish both (a) and the fact that the integrand on the right-hand side above is measurable?

2. Consider the following:

Theorem. *Let* $(X, \mathscr{M}, \mu)$ *and* $(Y, \mathscr{N}, \nu)$ *be* σ*-finite. If* $E \in \mathscr{M} \otimes \mathscr{N}$*, then the functions*

$$X \ni x \longmapsto \nu(E_x) \quad \text{and} \quad Y \ni y \longmapsto \mu(E^y)$$

are measurable and

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y).$$

In class, we: (i) **rigorously** proved the above theorem under the assumption that μ and ν are finite, and (ii) discussed how to use the hypothesis of σ-finiteness. Using the outcomes of (i) and (ii), complete the proof of the above theorem when $\mu(X) = \nu(Y) = \infty$.

3. Let $(X, \mathcal{F}, \mu)$ be a measure space, and suppose $f_1, \dots, f_N : X \longrightarrow \mathbb{F}$ are measurable functions, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $p_1, \dots, p_N \in [1, +\infty)$ such that $(1/p_1) + \dots + (1/p_N) = 1$. Show that

$$\|f_1 \dots f_N\|_1 \leq \prod_{j=1}^N \|f_j\|_{p_j}.$$

Note. We exclude $p_j = +\infty$ above **merely** so that one can give a cleaner argument. The inequality above would continue to hold if some of the p_j's above were equal to $+\infty$.

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a (Lebesgue) measurable function and suppose f is Lebesgue integrable on $(-1, 1)$. Let $Y \subsetneq \mathbb{R}$ be a closed subset. Assume that the *averages* of f on E,

$$\langle f \rangle_E := \frac{1}{m(E)} \int_E f\, dm$$

lie in Y for every $E \in \mathscr{M}|_{(-1,1)}$ for which $m(E) > 0$. Deduce that $f(x) \in Y$ for a.e. $x \in (-1, 1)$.
Note. This was one of the **relatively** easy (!) problems in the mid-semester exam that nearly everyone ignored. To solve this problem, please do **not** think complicatedly!

5. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of $[0, +\infty)$-valued measurable functions on X. Suppose $f : X \longrightarrow [0, +\infty)$ is a measurable function such that $f_n \xrightarrow{\mu} f$. Then, show that

$$\liminf_{n \to \infty} \int_X f_n \, d\mu \geq \int_X f \, d\mu.$$