MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: MARCH 28, 2023**

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1. Let $(X, \mathcal{F})$ be a measurable space and let ν be a complex measure on it. Show that $\mathrm{Re}(\nu)$ and $\mathrm{Im}(\nu)$ are signed measures.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\nu : \mathcal{F} \to [-\infty, +\infty]$ be a signed measure. Show that if $\nu \ll \mu$ as well as $\nu \perp \mu$, then ν is the identically-zero measure.

3. Let $(X, \mathcal{F})$ be a measurable space and let ν be a signed measure on it. Let (ν^+, ν^-) be a pair of measures such that

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-. \tag{1}$$

Let $(\widetilde{\nu}^+, \widetilde{\nu}^-)$ be a pair of measures satisfying (1) (with $\widetilde{\nu}^\pm$ replacing $\nu^\pm$). Let $A, B \in \mathcal{F}$ be such that $A \cap B = \varnothing$, $X = A \cup B$, A is a null set for $\widetilde{\nu}^+$ and B is a null set for $\widetilde{\nu}^-$. Show that $X = A \sqcup B$ is a Hahn decomposition of X and, thereby, show that $\widetilde{\nu}^+ = \nu^+$ and $\widetilde{\nu}^- = \nu^-$.

4. Let $F : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Show that $F(-\infty) := \lim_{x \to -\infty} F(x)$ exists.

5. Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded monotone function. Show that F is of bounded variation and give a formula for its total variation $V(F)$.

6. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a periodic function — i.e., $\exists T \in (0, +\infty)$ such that $F(x+T) = F(x)$ $\forall x \in \mathbb{R}$. When is F of bounded variation?

Note. This problem shows that being of bounded variation is quite a restrictive condition. While the class BV contains many functions that are not even continuous (e.g., see the previous problem), plenty of **very well-behaved** functions fail to be of bounded variation.

7. Let $F : \mathbb{R} \to \mathbb{R}$.

(*a*) Let $a < b \in \mathbb{R}$. Show that

$$V_F(b) - V_F(a) \;=\; \sup\left\{ \sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| : N \in \mathbb{Z}_+,\ a = x_0 < x_1 < \dots < x_N = b \right\}.$$

(*b*) Since the right-hand side of the above identity depends **only** on the values F takes on $[a, b]$:

- We call $\big(V_F(b) - V_F(a)\big)$ the *total variation of F on* $[a, b]$, which we denote by $V(F; [a, b])$.
- We say that a function $f : [a, b] \to \mathbb{R}$ is of bounded variation, denoted $f \in BV[a, b]$, if and only if $V(f; [a, b]) < \infty$.

Now define

$$F(x) \;:=\; \begin{cases} x \sin(1/x), & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0, & \text{if } x = 0. \end{cases}$$

Does $F|_{[0,\,b]}$ belong to $BV[0, b]$? Please **justify** your answer.

8. Let $(X, \mathcal{F})$ be a measurable space and let ν be a signed measure on it. Let $\nu_\nu^\pm$ be the (positive) measures associated to ν given by its Jordan decomposition. Consider the positive measure

$$|\nu| := \nu^+ - \nu^-.$$

Let $f : \mathcal{F} \to \mathbb{R}$ be a measurable function. Show that $f \in \mathbb{L}^1(\nu)$ if and only if $f \in \mathbb{L}^1(|\nu|)$.

9. Let $(X, \mathcal{F})$ be a measurable space and let ρ and τ be **finite** measures on it. Show that either $\rho \perp \tau$ or there exists a $\delta > 0$ and a set $A \in \mathcal{F}$ such that $\tau(A) > 0$ and

$$\rho(E) \geq \delta\tau(E) \;\; \forall E \subseteq A : E \in \mathcal{F}.$$

10. Let $(X, \mathcal{F}, \mu)$ be a measure space. **Assuming** (without proof) that the statement of the Lebesgue Decomposition Theorem is true under the additional assumption that ν — as appearing in the the statement of that theorem — is a (positive) measure, prove the general statement of the Lebesgue Decomposition Theorem.