MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 11

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Note: In what follows, if $(X, \mathcal{F}, \mu)$ is a measure space, then $\mathbb{L}^p(\mu)$ — without mention of the underlying field — will denote the $\mathbb{L}^p$-space arising from $\mathbb{R}$-valued measurable functions. An analogous explanation applies to $\mathbb{L}^1(\nu)$ for signed measures ν on $\mathcal{F}$.

1. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\nu : \mathcal{F} \to [-\infty, +\infty]$ be a σ-finite signed measure on $\mathcal{F}$ such that $\nu \ll \mu$. One of the reasons for the notation $(d\nu/d\mu)$ for the measurable function $f : X \to [-\infty, +\infty]$ such that $d\nu = f\,d\mu$ is the following identity

$$\int_X g\,d\nu \;=\; \int_X g\left(\frac{d\nu}{d\mu}\right)\,d\mu \tag{1}$$

for **every** $g \in \mathbb{L}^1(\nu)$, which is reminiscent of the change-of-variable formula of calculus in the Leibnizian notation. Prove (1) for $g = \chi_E$, where $E \in \mathcal{F}$ such that $\chi_E \in \mathbb{L}^1(X, \nu)$.

Remark. The above conclusion suffices — thanks to the definition and linearity of the Lebesgue integral — to establish (1) in general.

2. Let $(X, \mathcal{F})$ be a measurable space and let

$$M(\mathcal{F}) \;:=\; \{\nu : \mathcal{F} \to \mathbb{R} \mid \nu \text{ is a signed measure and } \|\nu\| < \infty\},$$

where we define $\|\nu\| := |\nu|(X)$. Show that $\|\boldsymbol{\cdot}\|$ is a norm on $M(\mathcal{F})$.

3. Define $\mathcal{C}_0(\mathbb{R}^n; \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, to be the closure of $\mathcal{C}_c(\mathbb{R}^n; \mathbb{F})$ with respect to the metric induced by the $\mathbb{L}^\infty$-norm. Show that

$$\mathcal{C}_0(\mathbb{R}^n; \mathbb{F}) \;=\; \big\{f : \mathbb{R}^n \to \mathbb{F} \mid f \text{ is continuous and } \lim_{\|x\| \to +\infty} f(x) = 0\big\}$$

Review problems. The following problems are drawn from across the syllabus of MA222.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable 1-periodic function (i.e., $f(x+1) = f(x)$ for every $x \in \mathbb{R}$). Suppose $\exists C \in (0, \infty)$ such that

$$\int_{[0,1]} |f(a+x) - f(b+x)|\,dm(x) \;\leq\; C \quad \forall a, b \in \mathbb{R}.$$

Show that $f|_{(0,1)}$ is Lebesgue integrable on $(0,1)$.

5. Consider the measure space $([0, 2\pi], \mathscr{M}_1|_{[0,2\pi]}, m|_{[0,2\pi]})$ and fix a function f in $\mathbb{L}^2\left(m|_{[0,2\pi]}\right)$. Does either of the sequences

$$\left\{\int_{[0,2\pi]} f(x)\cos(nx)\,dm\right\},$$

$$\left\{\int_{[0,2\pi]} f(x)\sin(nx)\,dm\right\}$$

converge? If so, then determine the limit — giving complete justifications.

6. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of $[0, +\infty)$-valued measurable functions on X. Suppose $f : X \to [0, +\infty)$ is a measurable function such that $f_n \xrightarrow{\mu} f$. Then, show that

$$\liminf_{n\to\infty} \int_X f_n \, d\mu \;\geq\; \int_X f \, d\mu.$$

7. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{A_n\}$ be a sequence in $\mathcal{F}$. Suppose the series

$$\sum_{n=1}^{\infty} \mu(A_n)$$

converges. Then, show that for μ-a.e. $x \in X$, x belongs to at most finitely many of the E_j's.

8. Let $(X, \mathcal{F}, \mu)$ be a measure space, and suppose $f_1, \dots, f_N : X \to \mathbb{F}$ are measurable functions, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $p_1, \dots, p_N \in [1, +\infty)$ such that $(1/p_1) + \dots + (1/p_N) = 1$. Show that

$$\|f_1 \dots f_N\|_1 \;\leq\; \prod_{j=1}^{N} \|f_j\|_{p_j}.$$

Note. We exclude $p_j = +\infty$ above **merely** so that one can give a cleaner argument. The inequality above would continue to hold if some of the p_j's above were equal to $+\infty$.

9. Let $a < b \in [-\infty, +\infty]$ and let $g : (a, b) \to \mathbb{R}$ be a convex function: which means that

$$g((1-t)x + ty) \;\leq\; (1-t)g(x) + tg(y) \quad \forall x, y \in (a, b) \text{ and } \forall t \in [0, 1].$$

(*a*) Show that g is (Lebesgue) measurable.

(*b*) Let $(X, \mathcal{F}, \mu)$ be a measure space, and suppose $\mu(X) = 1$. Let $f : X \to (a, b)$ belong to $\mathbb{L}^1(\mu)$. Show that $g \circ f$ is Lebesgue integrable in the extended sense and that

$$g\left(\int_X f \, d\mu\right) \;\leq\; \int_X (g \circ f) \, d\mu.$$

Hint. What property of g, once demonstrated, would **simultaneously** establish both (*a*) and the fact that the integrand on the right-hand side above is measurable?

10. Fix $n \in \mathbb{Z}_+ \setminus \{1\}$. Let $T \in GL(n, \mathbb{R})$. Let $E \in \mathscr{M}_n$ and let $f : E \to \mathbb{R}$ be measurable with respect to $\mathscr{M}_n|_E$.

(*a*) Argue, using the change-of-variable formula on $\mathbb{R}^n$, that $T^{-1}(E) \in \mathscr{M}_n$.

(*b*) Show that $f \circ T : T^{-1}(E) \to \mathbb{R}$ is measurable with respect to $\mathscr{M}_n|_{T^{-1}(E)}$.

(*c*) If $f \in \mathbb{L}^1(m|_E)$, then, appealing the change-of-variable formula on $\mathbb{R}^n$, show that $f \circ T \in \mathbb{L}^1\left(m|_{T^{-1}(E)}\right)$ and that

$$\int_E f \, dm \;=\; |\det(T)| \int_{T^{-1}(E)} f \circ T \, dm.$$