MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: JANUARY 18, 2023**

---

1. Let X be an uncountable set and define

$$\mathcal{F} := \{A \in \mathscr{P}(X) : \text{either } A \text{ or } X \setminus A \text{ is at most countable}\}.$$

Show that $\mathcal{F}$ is a σ-algebra.

2. Let $(X, \mathcal{F})$ be a measurable space and let $\mu : \mathcal{F} \longrightarrow [0, +\infty]$. Consider the following properties:

(m0) $\mu(\varnothing) = 0$;

(m1) For any $A_1, \dots, A_N \in \mathcal{F}$ that are pairwise disjoint $\mu\big(\cup_{1\leq j\leq N} A_j\big) = \sum_{1\leq j\leq N} \mu(A_j)$;

(m2) For any increasing sequence of sets $\{A_n\} \subset \mathcal{F}$, $\mu\big(\cup_{n=1}^{\infty} A_n\big) = \lim_{n\to\infty} \mu(A_n)$;

and a related property:

(M1) For any sequence of pairwise disjoint sets $\{A_n\} \subset \mathcal{F}$, $\mu\big(\cup_{n=1}^{\infty} A_n\big) = \sum_{n=1}^{\infty} \mu(A_n)$.

Show that μ has the properties (m0), (m1) and (m2) if and only if it has the properties (m0) and (M1).

3. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{A_n\} \subset \mathcal{F}$ be a decreasing sequence of sets. Assume that $\mu(A_1) < \infty$. Show that $\mu\big(\cap_{n=1}^{\infty} A_n\big) = \lim_{n\to\infty} \mu(A_n)$.

Give an example (consider $X = \mathbb{R}$) showing that the above conclusion is false if we drop the assumption that $\mu(A_1) < \infty$.

4. (Restriction of a measure) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $E \in \mathcal{F}$. Define $\mathcal{F}|_E := \{E \cap A : A \in \mathcal{F}\}$ and let $\mu|_E$ denote the restriction of μ to $\mathcal{F}|_E$. Show that $(E, \mathcal{F}|_E, \mu|_E)$ is a measure space.

5. Let $x_0 \in \mathbb{R}^n$ and let $\mathscr{B}(\mathbb{R}^n)$ denote the Borel σ-algebra on $\mathbb{R}^n$. Define $\mu : \mathscr{B}(\mathbb{R}^n) \to \{0, 1\}$ by

$$\mu(A) := \begin{cases} 1, & \text{if } x_0 \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Show that μ is a measure.

Remark. The above makes sense for any topological space X with the Borel σ-algebra on X replacing $\mathscr{B}(\mathbb{R}^n)$. The measure μ is called the *Dirac mass at* x_0.

6. Show that the Lebesgue outer measure m^* is translation invariant: i.e., that, fixing $n \in \mathbb{Z}_+$,

$$m^*(A) = m^*(x + A) \quad \text{for any } x \in \mathbb{R}^n \text{ and any } A \in \mathscr{P}(\mathbb{R}^n).$$

7. Fix a function $F : \mathbb{R} \to \mathbb{R}$ that is increasing (not necessarily strictly) and continuous from the **right.** For any $A \subseteq \mathbb{R}$, define

$$\mu_F^*(A) := \inf\Big\{ \sum_{j\in J}(F(b_j) - F(a_j)) : \{(a_j, b_j] : j \in J\} \text{ is an admissible cover of } A\Big\}.$$

Show that μ_F^* is an outer measure.

8. Fix $n \in \mathbb{Z}_+$. Let $\{E_j : 1 \leq j \leq N\}$ be a finite, pairwise disjoint collection of sets, where $E_1, \dots, E_N \in \mathscr{M}_n$. Show that

$$m^*\Big(\cup_{j=1}^{N}(A \cap E_j)\Big) = \sum_{j=1}^{N} m^*(A \cap E_j) \quad \forall A \in \mathscr{P}(\mathbb{R}^n).$$